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Measuring indecisiveness

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Abstract

We examine the problem of ordering decision-makers exhibiting incomplete preferences, by the extent of their decisiveness. We provide an axiomatic derivation of a new decisiveness ordering and two decisiveness metrics, and explore how these relate to one another, and to existing approaches in the literature. Illustrative examples are provided using consumer choice data.

Keywords Incomplete preferences, measurement of indecisiveness, set of order extensions of an incomplete preference preorder.

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1 Introduction

It has long been considered a postulate of rationality that a decision-maker has a preference relation that is transitive and complete over a choice set X. By *complete*, we mean that for any two alternative choices $x, y \in X$, the decision-maker either prefers x to y, y to x or is indifferent between x and y. A decision-maker who is sometimes unable to order choices is said to have *incomplete preferences*, or is said to be *indecisive*. There are many natural instances where in fact, it is far more realistic to assume the decision-maker to be indecisive. When choice involves options that are highly complex, or that the decision-maker is not sufficiently familiar with, it is natural for them to be indecisive, by postponing the decision for a later date (Gerasimou, 2018). If choice involves aggregating the preferences of several committee members, it is often likely that indecision arises due to lack of consensus. In coalition games, where a subset of agents may have a greater say over decision-making, or at the limit have veto power, indecisiveness is also very likely (a well-studied case is the decision making process of the United Nations' Security Council). If X is a set of social states, requiring the social planner to be endowed with a complete social welfare ordering would be a very demanding assumption, '*...for it is only natural to allow social ethics criteria not to be able to rank every social alternative.*' (Ok, 2002: p. 431). Choice under uncertainty may often result in indecision because the decision-maker cannot assess the exact probabilities of particular outcomes, or has no way of ascertaining payoffs, due to lack of familiarity with the particular risks involved.[1]

Alongside these theoretical settings where indecisiveness was shown to arise, there is growing experimental evidence that preferences are incomplete (see below). Thus, there is a need at the conceptual level to formalize different notions of indecisiveness, and at the empirical level, to synthesize the experimental evidence using appropriate tools of measurement, that is indecisiveness metrics and orderings. One may be tempted to use *height* and *width* measures already available in the context of *partial orders*, to measure incompleteness of preference *preorders*.[2] However, as the experimental

[1]For a theory of expected utility in the context of incomplete preferences, we refer the reader to Dubra, Maccheroni, and Ok (2004).

[2]We provide a formal definition of partial orders and preorders below. However, we mention here that a partial order is a special type of preorder that does not allow for decision-makers to exhibit indifference between distinct alternatives.

evidence documents, in the context of individual choice, agents experience separately phenomena of indifference and indecisiveness.[3] To account for both these phenomena, our interest in this paper will be in measuring incompleteness of preference preorders, as opposed to partial orders. Apart from Gorno (2018) who provides one such ordering, there are no measures that can easily be implemented. This paper introduces axioms for indecisiveness, and further axiomatically characterizes one new decisiveness order and two new decisiveness metrics. The latter are particularly convenient summary statistics to use in experimental work, and we illustrate the use of these new decisiveness measures in the context of preference preorders constructed by Gerasimou (2021) from consumer choice data.

The main results of the paper are the following. In Proposition 1 we axiomatically characterize the first decisiveness metric, defined as the number of pairs alternatives a decision-maker is able to compare from a finite choice set. The second decisiveness metric, the number of complete preorder extensions of an incomplete preference relation, is characterized in Proposition 2. En route, in order to obtain these characterization results, the paper also contributes to the theoretical indecisiveness literature from several perspectives. Firstly, we introduce the set of order extensions of an incomplete preference preorder as a general framework for examining models of indecisiveness in economics. In doing so, we generalize somewhat the set of order extensions of a partial order, initially introduced in the mathematical sciences by Brualdi, Jung, and Trotter (1994). To characterize the second decisiveness metric, we have found it also useful to introduce a novel definition of the decomposition of the preference relation of an indecisive decision-maker (Definition 3). This definition was used to obtain a novel parsimonious decomposition of the preference relation of an indecisive decision-maker (Lemma 1), that is somewhat computationally simpler to work with than the classic result of Szpilrajn (1930) and Richter (1966), defining an incomplete preference relation as the intersection of its set of complete preorder extensions. Further to characterising a function that counts the number of complete preorder extensions of an incomplete preference relation (Proposition 2), we also propose a simple method for ordering decision-makers according to the second decisiveness metric (Proposition 3). Such a method combines the new preorder decompo-

[3]For an extension of the axioms of revealed preference in the context where the decision-maker experiences indifference as well as indecisiveness, we refer the reader to Eliaz and Ok (2006).

sition along with some of the axioms that have been used to characterise the first decisiveness metric. Finally, we explore the formal relationship between the proposed decisiveness orderings (Corollary 2) by showing – amongst other things – that the two decisiveness metrics are logically independent.

This paper has arisen as a necessity to complement the experimental literature documenting the pervasiveness of indecisiveness, with an appropriate framework and tools of measurement. Danan and Ziegelmeyer (2006) and Cettolin and Riedl (2019) provide evidence that preferences are incomplete in experiments involving choices under risk and uncertainty, respectively. Qiu and Ong (2017) design an experiment aimed at disentangling indifference from indecisiveness in choice and find strong evidence in favour of the latter. Costa-Gomes, Cueva, Gerasimou, and Tejiščák (2022) and Gerasimou (2021) find that, forcing subjects to choose increases the extent to which their choice behavior becomes inconsistent. They report that a substantial fraction of subjects' decisions can be explained by preferences being incomplete. Although our work is theoretical, it nonetheless relates to the experimental literature in at least two ways. First, the decisiveness orderings studied in this paper can be employed to measure the decisiveness of preferences that are elicited in experimental studies. A simple illustration of such an exercise is given in Section 5 of this paper. Second, in experimental and – more generally – applied work that explores the relationship between decisiveness and other economically relevant variables, the orderings studies in this paper can be used to formulate proxies for an agent's degree of decisiveness.

Our work also relates to the theoretical literature on indecisiveness. Eliaz and Ok (2006) and Gerasimou (2018) both investigate the choice-theoretic implications of decision-makers being indecisive. While in the former, indecisiveness over a pair of alternatives x and y is revealed whenever decision-makers choose both x and y (and some other conditions are satisfied), in the latter indecisiveness is revealed whenever decision-makers defer the choice between x and y. Despite this difference, in both models the reason why the individual exhibits these choice behaviours is that they are unable to compare x and y. Although this paper investigates a different research question, the present paper relates to Eliaz and Ok (2006) and Gerasimou (2018), in that the first decisiveness metric that is axiomatically characterised in this paper uses the same principle that Eliaz and Ok (2006) and Gerasimou (2018) utilise to conceptualise indecisiveness, i.e., counting the number of instances in which a decision-maker is unable to compare two alternatives.

On the other hand, Gorno (2018) studies the structure of incomplete preferences by proposing one decisiveness order. Our work complements Gorno (2018) by introducing one further decisiveness ordering and two decisiveness metrics and by further exploring the relationships between them.

One further strand of theoretical literature on indecisiveness investigates the multi-utility representation of incomplete preferences (Ok, 2002). In that literature, a key result that is used to derive a multi-utility representation is that the intersection of all the complete preorder extensions that extend a given incomplete preference relation is equal to the incomplete preference relation itself (Szpilrajn, 1930; Richter, 1966). Although this paper studies a different problem, it relates to this literature in that the second decisiveness metric that is axiomatically characterised in this paper is based on the very same result, in that it counts the number of complete preorder extensions of an incomplete preference relation.

The next section introduces *the set of preorder extensions* of an incomplete preference relation. There, we also define a new decisiveness order and two related metrics on the set of preorder extensions, that we shall axiomatize in subsequent sections of the paper. In Section 3 we introduce a number of axioms for decisiveness order relations, and characterize the first decisiveness metric. In Section 4, we introduce further axioms, and a novel decomposition of preference preorders, that are further used in order to characterize the second decisiveness metric. In Section 5, we apply these decisiveness orders and metrics in the context of preference preorders recovered by Gerasimou (2021) in the context of consumer choice. Section 6 concludes the paper. The appendix contains proofs of all results, and further demonstrates that the axioms introduced in this paper are logically independent.

2 Preorders, order extensions, and decisiveness

We begin this section with a number of definitions and properties of preordered sets. We then specialize this discussion in the context of choice over a finite set, where we introduce the set of order extensions of an incomplete preference preorder. We then introduce several decisiveness metrics defined on the set of preorder extensions.

Let X be an n-element set. To simplify the discussion in this paper, we shall assume throughout that the indecisive decision-maker is endowed with a preference relation $\succeq$ over X, that is reflexive and transitive. We denote the resulting preorder $E := (X, \succeq, \succ)$, where the notation $x \succ y$ denotes that $x \succeq y$ while $y \not\succeq x$. We also let $x \sim y$ signify that $x \succeq y$ and $y \succeq x$, so that $\succ$ is an asymmetric relation, while $\sim$ is symmetric. Two distinct elements x and y of X are said to be *comparable* if either $x \succeq y$ or $y \succeq x$. The notation $x \mid\mid y$ is used to denote that x and y are not comparable. The set of incomparable elements associated with the preorder $(X, \succeq, \succ)$ is defined as $inc(X, \succeq, \succ) := \{(x, y) \in X \times X : x||y\}$. The set of *comparable pairs* or *comparability set* is defined as $comp(X, \succeq, \succ) := (X \times X) \backslash inc(X, \succeq, \succ)$, and we sometimes use the simpler notation $comp(E)$ for the comparability set of $(X, \succeq, \succ)$. When every pair $(x, y) \in X \times X$ is $\succeq$-comparable, we say that $(X, \succeq, \succ)$ is *a complete preorder*. When $comp(E) = \emptyset$, we say that the decision-maker is completely indecisive. Accordingly, their preferences are structured as an *antichain*.[4]

Example 1. Consider a four-element choice set $X = \{a, b, c, d\}$ and two preference preorders that we denote $E_1 = (X, \succeq^1, \succ^1)$ and $E_2 = (X, \succeq^2, \succ^2)$. We assume that $a \sim b \sim c$ in the relation E_1 and furthermore that d is incomparable to a, b and c. Because a preorder is reflexive, every element is indifferent to itself. For compactness, we omit writing the reflexive part of the relation and we simply write $\succeq^1 = \{(a, b), (a, c), (b, c), (b, a), (c, a), (c, b)\}$, while $\succ^1$ is equal to the empty set. We assume that $\succ^2 = \{(a, b), (a, c), (b, c), (a, d)\}$ and $\succeq^2 = \succ^2$.$\square$

It is convenient in the context of finite preorders to depict the relation graphically in terms of a Hasse diagram. In Figure 1, we sketch the Hasse diagrams for the preorders E_1 and E_2. We also define the incomparability sets of the two relations, as follows: $inc(X, \succeq^1, \succ^1) = \{(a, d), (b, d), (c, d)\}$ and $inc(X, \succeq^2, \succ^2) = \{(b, d), (c, d)\}$. From this information we can also deduce the comparability sets. They are given by $comp(X, \succeq^1, \succ^1) = \{(a, b), (a, c), (b, c)\}$ and $comp(X, \succeq^2, \succ^2) = \{(a, b), (a, c), (b, c), (a, d)\}$.

[4]We define a relation $(X, \succeq^1, \succ^1)$ on a finite choice set to have the structure of an antichain whenever, for all $a, b \in X$, a and b are incomparable.

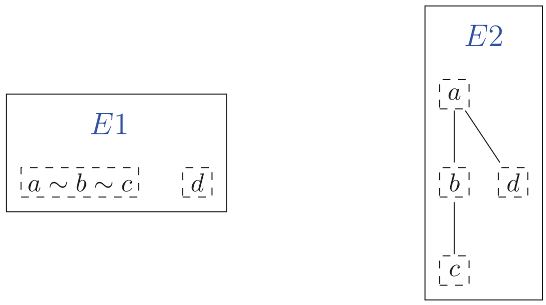

Figure 1: Hasse diagrams of preorders E_1 and E_2

2.1 The set of order extensions of an incomplete preference preorder

Our interest in this paper being in measuring indecisiveness of decision-makers, we next introduce the *set of preorder extensions* of an *antichain* defined on a finite choice set. We first define the concept of an extension of a preorder (see Chambers and Echenique (2016: p. 5) for a general discussion).

Definition 1 (Preorder Extension). *Let $E_i := (X, \succeq^i, \succ^i)$ and $E_j := (X, \succeq^j, \succ^j)$ denote two preorders on a finite choice set X. We shall say that E_j is a preorder extension of E_i, if $\succeq^i$ is a subset of $\succeq^j$ and $\succ^i$ is a subset of $\succ^j$. We denote that E_j is an extension of E_i by $E_j \succeq_{d^{ext}} E_j$.*

More simply E_j is a preorder extension of E_i whenever, if decision-maker i regards tea to be at least as good as coffee, then decision-maker j likewise perceives tea to be at least as good as coffee, and if decision-maker i strictly prefers chocolate ice-cream to vanilla ice-cream, then decision-maker j also strictly prefers chocolate over vanilla ice-cream.[5] In other words, preorder E_j is a preorder extension of preorder E_i whenever E_j orders the pairs of alternatives in X in the same way as E_i does, and possibly orders additional pairs of alternatives. As such, E_i is always an order extension of itself.

For a fixed choice set X and a relation $E_i := (X, \succeq^i, \succ^i)$, we may define a set that contains *all* preorder extensions of E_i using the following notation:

[5]Observe that if $E_j \succeq_{d^{ext}} E_i$, Definition 1 prohibits that $a \sim^i b$ while $b \succ^j a$, and likewise it is not possible that $b \succ^i a$ while $a \sim^j b$.

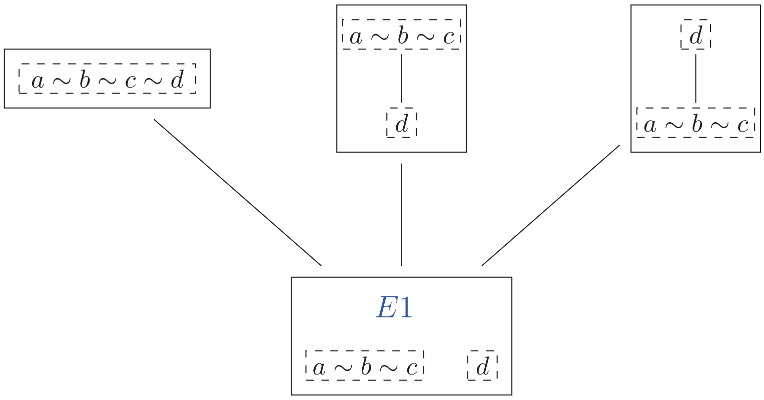

Figure 2: The set of preorder extensions of E_1

$$\uparrow E_i := \{E_j : E_j \succeq_{d^{ext}} E_i\}. \tag{1}$$

If the preorder E_i is incomplete, then there are clearly diverse ways in which the indecisiveness in E_i can be resolved. The set $\uparrow E_i$ describes all possible manners in which preorder E_i can be completed, by considering the different ways in which the pairs of alternatives in the incomparability set $\text{inc}(X, \succeq^i, \succ^i)$ of E_i can be ordered. In Figures 2 and 3, we depict the sets $\uparrow E_1$ and $\uparrow E_2$ of Example 1. Observe that in Figure 2 depicting $\uparrow E_1$, a relation E_j is an order-extension of E_i when E_j appears above E_i, and additionally when there is a sequence of edges that connects the two relations. Observe that the relation $\succeq_{d^{ext}}$ is reflexive, antisymmetric, and transitive over a given set $\uparrow E$. As such, each of the ordered pairs $(\uparrow E_1, \succeq_{d^{ext}})$ and $(\uparrow E_2, \succeq_{d^{ext}})$ is a *partial order*,[6] and the diagrams of Figures 2 and 3 are Hasse diagrams. It important furthermore to note that the relation $\succeq_{d^{ext}}$ is not complete. In Figure 3, for example, the left-most preorder on the second level and the right-most preorder in the third level are not comparable by $\succeq_{d^{ext}}$; in the former $b \succ d$, in the latter $b \sim d$.

In this paper, the set of preorder extensions of an n-element antichain will play a prominent role. Using Definition 1, we define $E_0 := (X, \succeq^0, \succ^0)$ such that $\succeq^0$ and $\succ^0$ are both empty sets. Because E_0 has the structure

[6] A partial order is a preorder where for all $a, b \in X$, $a \succeq b$ and $b \succeq a$, implies that $a = b$. That is, a partial order is an antisymmetric type of preorder.

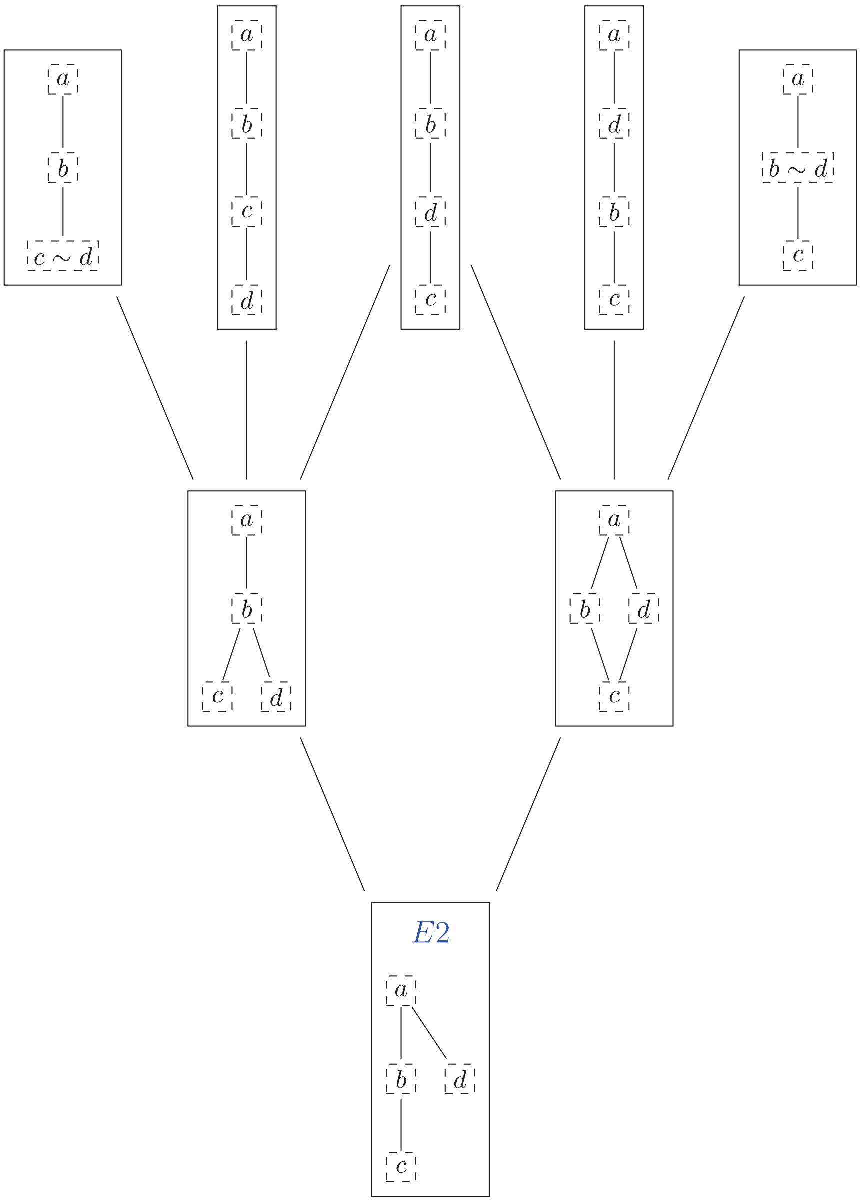

Figure 3: The set of of preorder extensions of E_2

of an n-element antichain, we refer to E_0 as the preorder associated with the *completely indecisive* decision-maker. We next introduce the set of order extensions of the preference preorder E_0, associated with the completely indecisive decision-maker:

$$\uparrow E_0 = \{E_j : E_j \succeq_{d^{ext}} E_0\} \tag{2}$$

By analogy with the above discussion, it follows that $(\uparrow E_0, \succeq_{d^{ext}})$, the set of preorders that extend the n-element antichain, is structured as a partial ordering, with the property that $E_i := (X, \succeq^i, \succ^i) \succeq_{d^{ext}} E_j := (X, \succeq^j, \succ^j)$ if E_j is a more complete preorder than E_i. Observe that $(\uparrow E_0, \succeq_{d^{ext}})$ is *exhaustive,* in the sense that any preorder defined on the choice X must be an element of the set of preorder extensions. It is for this reason that we shall define decisiveness orders on $(\uparrow E_0, \succeq_{d^{ext}})$ below.

Note also that the set

$$ext(E_0) := \{E \in \uparrow E_0 : E \text{ is a partial order}\} \tag{3}$$

defines the set of *poset* extensions of the antichain E_0. Given that ext(E_0) $\subseteq \uparrow E_0$, the set of preorder extensions defined in this paper generalises the set of poset extensions, that has been extensively investigated in Brualdi, Jung, and Trotter (1994).[7]

Following theorems of Szpilrajn (1930) and Richter (1966), for every preorder $E_i \in \uparrow E_0$, there exists a complete preference relation in $\uparrow E_0$ that extends E_i. It follows accordingly that the maximal elements of $(\uparrow E_0, \succeq_{d^{ext}})$ are complete preference relations. For a given preorder E, we denote the subset of complete preorder extensions of E by $\mathcal{C}(E)$, where

$$\mathcal{C}(E) := \{F \in \uparrow E : F \text{ is a complete preorder}\} \tag{4}$$

In figure 3 the set $\mathcal{C}(E_2)$ of complete preorder extensions of E_2 therefore consists of five preference relations. In the special case where E is the preference relation of the completely indecisive decision-maker, i.e. $E = E_0$, $\mathcal{C}(E_0)$

[7]Brualdi, Jung, and Trotter (1994) show that, within the set of poset extensions ext(E_0), every maximal chain is of equal length; that is, $(\text{ext}(E_0), \succeq_{d^{ext}})$ is a *ranked* poset. In contrast, it turns out that the set of pre-order extensions $(\uparrow E_0, \succeq_{d^{ext}})$ defined in this paper is not ranked. To see why, take a three-element choice set $X = \{a, b, c\}$ and observe that (i) any maximal chain whose top element is the linear order $a \succ b \succ c$ is of length four, and (ii) any maximal chain whose top element is the preorder $a \sim b \sim c$ is of length three.

is the set of *all* complete preference preorders. In our illustrative examples below, we shall provide examples of $\uparrow E_i$ for several preorders E_i constructed by Gerasimou (2021) from consumer choice data.

2.2 Measuring indecisiveness on the set of preorder extensions

We can observe from figure 3 that preference relations that are located closer up to the maximal elements of the set of preorder extensions tend to exhibit more decisiveness than those positioned further down in the set of preorder extensions. We now introduce the decisiveness orders and metrics we shall be defining on the set of preorder extensions $(\uparrow E_0, \succeq_{d^{ext}})$. We can think of each of the decisiveness orders of Definition 2 below as conceptualising the above intuition in different ways.

Definition 2 (Four Decisiveness Orders). *Let $E_i = (X, \succeq^i, \succ^i)$ and $E_j = (X, \succeq^j, \succ^j)$ denote two relations in the set of preorder extensions $\uparrow E_0$. We define E_i to be less decisive than E_j in the following four ways:*

(i) $E_j \succeq_{d^{ext}} E_i$, *equivalently,* $E_j \in \uparrow E_i$,

(ii) $E_j \succeq_{d^{comp}} E_i$ *if the comparability set* $comp\,(E_i)$ *is a subset of* $comp\,(E_j)$,

(iii) $E_j \succeq_{d^{card-comp}} E_i$ *if* $|comp\,(E_j)\,| \geq |comp\,(E_i)\,|$; *that is if* $comp\,(E_i)$ *has fewer elements than* $comp\,(E_j)$,

(iv) $E_j \succeq_{d^{up}} E_i$ *if* $|\mathcal{C}(E_i)| \geq |\mathcal{C}(E_j)|$; *that is, if* E_i *has more complete preorder extensions than* E_j.

The first decisiveness relation $\succeq_{d^{ext}}$ is due to Gorno (2018). The next three relations are new. The decisiveness relation $\succeq_{d^{ext}}$ is very natural: decision-maker j is taken to be more decisive than decision-maker i whenever E_j is a preorder extension of E_i in the sense of Definition 1. The second relation $\succeq_{d^{comp}}$ ranks decision-maker j to be more decisive than decision-maker i whenever decision-maker j is able to order the same pairs of alternatives as those of decision-maker i, and possibly is able to order additional pairs. Unlike $\succeq_{d^{ext}}$, $\succeq_{d^{comp}}$ does not consider how the pairs of alternatives are ordered, but only whether they are comparable. On the other hand, as in the case of $\succeq_{d^{ext}}$, $\succeq_{d^{comp}}$ is an incomplete relation.[8] [9]

We next turn our attention to the two decisiveness metrics. The third relation, $\succeq_{d^{card-comp}}$, ranks decision-maker j to be more decisive than decision-maker i whenever decision-maker i is able to compare fewer pairs of alternatives than decision-maker j. The choice-theoretic literature indicates that indecisiveness over a pair of alternatives x and y is revealed whenever decision-makers either (i) choose both x and y (and some other conditions are satisfied) (Eliaz and Ok, 2006), or (ii) defer the choice between x and y (Gerasimou, 2018). In these models, the reason why the individual exhibits these choice behaviours is that they are unable to compare x and y. As such, according to the metric $\succeq_{d^{card-comp}}$ an individual i is taken to be more decisive than some other individual j, whenever there are fewer instances whereby individual i is unable to compare a pair of alternatives x and y relative to individual j.

Finally, recall that the set $\mathcal{C}(E_i)$ contains all the different complete preorders that extend the preferences of decision-maker i. As such, if the set $\mathcal{C}(E_j)$ contains fewer preference relations than $\mathcal{C}(E_i)$, according to the fourth criterion, $\succeq_{d^{up}}$, decision-maker j is taken to be more decisive than i. Consider two individuals, i and j, with incomplete preferences given by E_i and E_j, respectively. Suppose that their preferences satisfy the property that $|\mathcal{C}(E_i)| \geq |\mathcal{C}(E_j)|$. This means that for individual i there is a greater number of complete preference preorders that complete their preferences. As such, one can view the task of resolving indecisiveness to be more difficult for individual i than for individual j, precisely because the former individual has more degrees of freedom than the latter - in the sense of being confronted with a larger number of possibilities to complete their preferences. Given that difficulty of selection is typically associated with decision avoidance and indecisiveness (Anderson, 2003), individual j is defined to be more decisive

[8]To see why, consider a three-element choice set $X = \{a, b, c\}$ and two preorders on X: (i) a preorder whereby $a \sim b$ and both a and b are incomparable to c and (ii) a preorder whereby $a \sim c$ and both a and c are incomparable to b. Clearly, these two preorders are not comparable by $\succeq_{d^{comp}}$.

[9]One distinction to make between the two decisiveness relations is that while $(\uparrow E_0, \succeq_{d^{ext}})$ is a partial order, the latter, $(\uparrow E_0, \succeq_{d^{comp}})$, is a preorder. In other words, in the context of $\succeq_{d^{ext}}$, if decision-maker i is more decisive than j and likewise j is more than decisive than i, the two decision-makers necessarily have identical preferences. In contrast, this property does not hold in the context of the latter decisiveness relation.

than individual i according to $\succeq_{d^{up}}$.

Our next task is to provide an axiomatic foundation for the various definitions of decisiveness introduced in this paper.

3 Characterisation of a First Measure of Decisiveness - $d^{card-comp}$

We begin the task of characterising the decisiveness metric $d^{card-comp}$ by introducing a number of axioms.

Let $\succeq_d$ denote a decisiveness relation on the set of preorder extensions $\uparrow E_0$. We consider the following properties.

Transitivity [A1]. $\succeq_d$ is a transitive relation.

Transitivity [A1] is clearly a natural property that defines any order relation and, more so, in the context of a decisiveness metric.

The next axiom [A2] is a rewriting of Definition 1.

Order Extension [A2]. For all E_i and E_j in the set of preorder extensions $\uparrow E_0$, if E_j is a preorder extension of E_i, then $E_j \succeq_d E_i$.

Axiom [A2] ensures that, if preorder E_j is an order extension of E_i, then the decisiveness metric $\succeq_d$ ranks E_j to be more decisive than E_i. Return to Figure 3 and observe there that the diamond-shaped preorder is an order extension of E_2. Axiom [A2] then requires that the diamond-shaped preorder is more decisive than E_2.

Comparability-Graph Invariance [A3]. For all E_i and E_j in the set of preorder extensions $\uparrow E_0$, if E_i and E_j have identical comparability graphs, then $E_i \sim_d E_j$.

Axiom [A3] postulates that if two preorders have identical comparability sets, then they should be considered to be as decisive as one another according to relation $\succeq_d$. For instance, in Figure 4 E_i and E_j are distinct preference relations that have identical comparability sets. According to [A3], we take E_i and E_j to equally decisive.

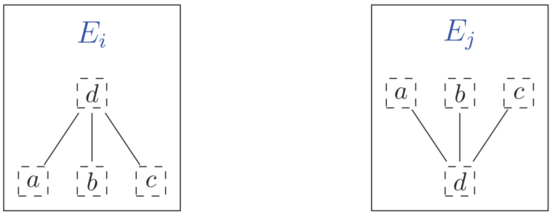

Figure 4: An Illustration of the Comparability-Graph Invariance Axiom [A3]

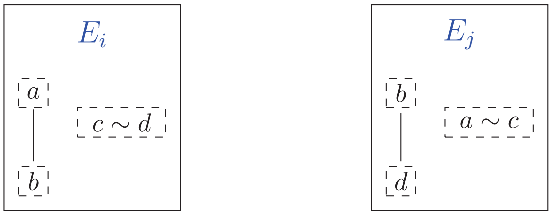

Figure 5: An Illustration of the Hasse-Diagram Invariance Axiom [A4]

Hasse-Diagram Invariance [A4]. For all E_i and E_j in the set of preorder extensions $\uparrow E_0$, if the Hasse diagrams of E_i and E_j are identical up to a relabelling of vertices, then $E_i \sim_d E_j$.

Axiom [A4] is best explained with the help of Figure 5. There E_i and E_j have identical Hasse diagrams, up to a permutation of their vertices. In E_i, $a \succ b$ and $c \sim d$. On the other hand, in the preference relation E_j, $b \succ d$ and $a \sim c$. Because the Hasse diagram of E_j represents a preference relation obtained from E_i via the vertex permutation $\{a, b, c, d\} \mapsto \{b, d, c, a\}$, axiom [A4] requires that E_i and E_j are equally decisive.

Before we introduce our next axiom, we need to introduce further notation as well as the concept of a *passive pair*. For a preference preorder $E_i = (X, \succeq^i, \succ^i)$ with the property that a given pair $(x, y) \notin comp(E_i)$, we denote by $E_i \cup \{x \succ y\}$ the *binary relation* that results from appending to both $\succeq^i$ and $\succ^i$ the ordered pair $x \succ y$. Observe that we refer to $E_i \cup \{x \succ y\}$ only as a

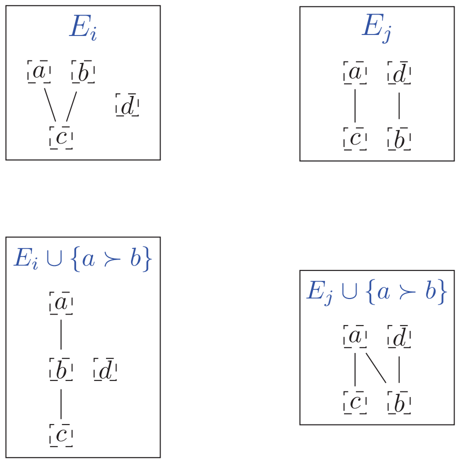

Figure 6: An Illustration of the Independence Axiom [A5]

binary relation, because this relation may fail to be transitive.[10] Next, let E_i be an incomplete preorder in $\uparrow E_0$, and let (x, y) be an incomparable pair at E_i. Following Caspard, Leclerc, and Monjardet (2012), we say that $x \succ y$ is a *passive pair* for E_i whenever $E_i \cup \{x \succ y\}$ is still a preorder.[11]

Independence [A5]. For all E_i and E_j in the set of preorder extensions $\uparrow E_0$ such that

(i) $(x \succ y) \notin E_i, E_j$;

(ii) $x \succ y$ is a passive pair for both E_i and E_j.

If $E_i \sim_d E_j$, then $E_i \cup \{x \succ y\} \sim_d E_j \cup \{x \succ y\}$.

[10] We provide in Figure 6 two examples of preorders E_i and E_j to which we append a pair $a \succ b$, which initially did not belong to their comparability sets, with the result that $E_i \cup \{a \succ b\}$ and $E_j \cup \{a \succ b\}$ are transitive relations.

[11] See in particular Definition 1.34 of page 21 in Caspard, Leclerc, and Monjardet (2012), which is given in the context of partial orders. Observe that in the more general context of preorders, $x \sim y$ may also define a passive pair. We return to this point further down in the paper. It is also common usage to refer to a passive pair under the name of a *critical pair*.

Axiom [A5] requires that, if two incomplete preference relations E_i and E_j are equally decisive according to $\succeq_d$ and $a \succ b$ is a passive pair for both E_i and E_j, then adding this same passive pair to both preference relations renders $E_i \cup \{a \succ b\}$ and $E_j \cup \{a \succ b\}$ equally decisive according to $\succeq_d$. We illustrate this axiom in Figure 6. In this example, $(a \succ b)$ is a passive pair in both E_i and E_j. Accordingly, if $E_i \sim_d E_j$, axiom [A5] ensures that in the bottom panel $E_i \cup \{a \succ b\}$ and $E_j \cup \{a \succ b\}$ must remain equally decisive.

Before we present the main result of this section, we note an intermediate result that is obtained by combining the first three axioms discussed above. Recall that $E_i \succeq_{d^{comp}} E_j$ if the comparability set of E_j is a subset of $comp(E_i)$. The decisiveness relation $\succeq_{d^{comp}}$ can immediately be seen to be characterised by axioms [A1], [A2], and [A3]. Note that transitivity is needed: while [A3] enables the construction of indifference classes of $\succeq_{d^{comp}}$ and [A2] enables the construction of the strict subrelation $\succ_{d^{comp}}$, transitivity [A1] is required to characterise the transitive closure of these two components of $\succeq_{d^{comp}}$. On the other hand, by definition, the decisiveness relation $\succeq_{d^{ext}}$ is characterised entirely by [A2]. We gather these observations in the following remark.

Remark 1.

- *A decisiveness measure $\succeq_d$ is equal to $\succeq_{d^{comp}}$ if and only if it satisfies [A1]+[A2]+[A3].*
- *A decisiveness measure $\succeq_d$ is equal to $\succeq_{d^{ext}}$ if and only if it satisfies [A2].*

The next result states that a decisiveness measure on the set of preorder extension is equal to $\succeq_{d^{card-comp}}$ if and only if it satisfies axioms [A1]-[A5].

Proposition 1 (Characterisation of $d^{card-comp}$)**.** *Let $\succeq_d$ denote some decisiveness relation on the set of preorder extensions $\uparrow E_0$. The relation $\succeq_d$ satisfies transitivity [A1], order-extension [A2], comparability-graph invariance [A3], Hasse-diagram invariance [A4], and independence [A5] if and only if $\succeq_d = \succeq_{d^{card-comp}}$.*

Although the proof of this result is detailed in the appendix, we here provide a summary of the three main steps involved. Firstly, Step 1 consists in defining a surjective map from the set of preorder extensions $\uparrow E_0$ to the set of poset extensions $ext(E_0)$, with the additional property that the map is order-preserving for the decisiveness relation. The order-preserving property is obtained by making use axiom [A3] together with transitivity [A1]. In the end of Step 1, it becomes possible to focus our attention in the proof on the relations defined in the set of poset extensions.

Our interest in Step 1 in moving from the set of preorder extensions to the set of poset extensions is to make use of the convenient property, that in the latter set, every maximal chain has identical length (footnote 7). As such, in Step 2 it becomes possible - within the set of poset extensions - to construct a proof by induction on the cardinality of the comparability sets. From the transitivity and Hasse-Diagram Invariance axioms, all posets with one comparable pair become equally decisive. From then on, we appeal to axiom [A5] to prove by induction that two posets E_i and E_j with an equal number of comparable pairs are equally decisive.

The final step consists of showing that if E_i has more comparable pairs than E_j, then - by using Step 2, transitivity [A1], and the order-extension property [A2] - E_i is more decisive than E_j.

Because in Step 1 we are able to map the set of preorder extensions to the set of poset extension using an order-preserving function for the decisiveness relation and Steps 2 and 3 no longer appeal to axiom [A3], it is possible to state the following corollary of Proposition 1.

Corollary 1. *Let $\succeq_d$ denote some decisiveness relation defined on the subset of poset extensions $ext(E_0) \subset \uparrow E_0$. The relation $\succeq_d$ satisfies transitivity [A1], order-extension [A2], Hasse-diagram invariance [A4], and independence [A5] if and only if $\succeq_d = \succeq_{d^{card-comp}}$.*

The decisiveness orders and metrics axiomatically characterised in this paper will be put to work on consumer data from the study of Gerasimou (2021) in Section 5 of the paper.

4 Characterisation of the Second Decisiveness Measure - d^{up}

Our next task will be to axiomatically characterise the second decisiveness metric of Definition 2, namely d^{up}. Recall that $E_i \succeq_{d^{up}} E_j$ whenever the cardinality of the set $\mathcal{C}(E_i)$ of complete preorder extensions of E_i is smaller than, or equal to, the cardinality of $\mathcal{C}(E_j)$. Our strategy here is somewhat different from the one adopted in the section above, in that - rather than directly axiomatising the relation $\succeq_{d^{up}}$ - we characterise a function $f : \uparrow E_0 \to \mathbb{N}$

that counts the number of complete preorder extensions of a given preference relation.

We begin with a novel preorder decomposition that we shall make use of repeatedly.

Definition 3 (Preorder Decomposition)**.** *Let $E \in\uparrow E_0$ be an incomplete preorder. A set of preorders $\langle P_1, \ldots, P_K \rangle$ is a* ***preorder decomposition*** *of E whenever $\langle \mathcal{C}(P_1), \ldots, \mathcal{C}(P_K) \rangle$ forms a partition of $\mathcal{C}(E)$.*

In the context of indecisiveness, the preorder decomposition captures the intuitive property that an indecisive decision-maker can be represented by a collection of their multiple selves $P_1, \ldots, P_K$ (Fudenberg and Levine, 2006). The definition formalises the properties the multiple selves must satisfy, by requiring that an incomplete preorder E is represented by a number of preorders, in the sense that the sets of complete preorder extensions $\mathcal{C}(P_1), \ldots, \mathcal{C}(P_K)$ partition the set of complete preorder extensions of E. One instance of this decomposition is given by the theorems of Szpilrajn (1930) for partial orders and Richter (1966); that is, the instance where $P_1, \ldots, P_K$ are complete preorder extensions of preorder E. A further instance of the decomposition is provided in Lemma 1 below.

Lemma 1. *Let $E \in\uparrow E_0$ be an incomplete preorder. Then, there are three preorders $P, Q, R \in\uparrow E_0$ such that $\langle P, Q, R \rangle$ is a preorder decomposition of E.*

The above result shows that with as few as three elements, it is possible to obtain a preorder decomposition in the sense of Definition 3. Note that this decomposition is parsimonious in the sense that it is not possible to have a smaller set of preorders that decompose an incomplete relation.[12] With the Decomposition Lemma it is now possible to state the axioms characterise to the decisiveness measure d^{up}.

Recall that the function f maps each given preorder to a natural number. The purpose of the axiomatisation exercise is to impose restrictions on the

[12]Consider the following simple case: $X = \{a, b\}$ and the preference relation E is given by the two-element antichain. Even in this simplest case, three relations are needed in order to obtain the preorder decomposition of E, namely, a preorder P_1 such that $a \succ b$, a preorder P_2 where $a \sim b$, and a preorder P_3 where $b \succ a$.

function f so that it counts the number of complete preorder extensions of a preorder E. We consider the following.

Additivity [B1]: Let $E, P, Q, R \in \uparrow E_0$ be preorders, where E is incomplete, and let $f : \uparrow E_0 \to \mathbb{N}$. If $\langle P, Q, R \rangle$ is a preorder decomposition of E, then $f(E) = f(P) + f(Q) + f(R)$.

Return to the multiple selves that decompose a preorder in the sense of Definition 3. Lemma 1 states that in general there is a decomposition with three such selves. Axiom [B1] then requires that the indecisiveness of a decision-maker is additive in the indecisiveness of the three such selves.

Normalisation [B2]: Let $G \in \uparrow E_0$ be a complete preorder. Then, $f(G) = 1$.

Axiom [B1] is a normalisation axiom that characterises the decision-makers on the set of preorder extensions that are completely decisive.

With the above axioms we are now ready to state our next result.

Proposition 2 (Characterisation of d^{up}). *Let $E \in \uparrow E_0$ be a preorder, and let $\mathcal{C}(E)$ denote the set of complete preorder extensions of E. Then, $f(E) = |\mathcal{C}(E)|$ if and only if $f : \uparrow E_0 \to \mathbb{N}$ satisfies Additivity [B1] and Normalisation [B2].*

Proposition 2 states that a mapping from the set of preorder extensions to the natural numbers is the decisiveness metric that counts the number of complete preorder extensions of E if and only if the mapping satisfies the additivity axiom [B1] as well as the normalisation property [B2]. In turn recall that the additivity axiom [B1] structures the set of preorders $\langle P, Q, R \rangle$ (multiple selves) over which the function f embodies the additivity property in the sense that $\langle P, Q, R \rangle$ must partition the set of complete preorder extensions of E (Definition 3 and Lemma 1). In Proposition 3 below, we shall formalise the fact that the function $f(E)$ provides a numerical representation of the relation $\succeq_{d^{up}}$.

The result is demonstrated using a proof by induction on the cardinality of the set of complete preorder extensions of an arbitrary preorder E. We start the induction using the normalisation axiom [B2], where we exploit the property that $|\mathcal{C}(E)| = 1$ for any complete preorder E. On the other hand, when E is an incomplete relation, we repeatedly use Lemma 1 together with the additivity axiom [B1], in order to arrive at the result.

Observe that the decisiveness metric $\succeq_{d^{up}}$ satisfies transitivity [A1], order-extension [A2], and Hasse-diagram invariance [A4]. As a further application of Lemma 1 and the above proposition, we state the following result.

Proposition 3. *Assume that $f :\uparrow E_0 \to \mathbb{N}$ satisfies axioms [B1] and [B2]. Then, the following statements are equivalent.*

(i) $E_j \succeq_{d^{up}} E_i$

(ii) $f(E_j) \leq f(E_i)$

(iii) There exist preorder decompositions $\langle P_1, \ldots, P_K \rangle$ and $\langle Q_1, \ldots, Q_L \rangle$ of E_i and E_j, respectively, such that:

- *(a)* $K \geq L$
- *(b) for each $h \in \{1, \ldots, L\}$, either $Q_h \succeq_{d^{ext}} P_h$ up to a relabelling of vertices, or P_h and Q_h are both complete preorders.*

In accordance with Proposition 2, items (i) and (ii) of Proposition 3 formalise the fact that a mapping from the set of preorder extensions to the natural numbers, that satisfies [B1] and [B2], is a numerical representation of the decisiveness metric $\succeq_{d^{up}}$ introduced in Definition 2. In turn, from (iii) the statement that E_j is more decisive than E_i according to $\succeq_{d^{up}}$ is equivalent to the existence of preorder decompositions $\langle P_1, \ldots, P_K \rangle$ and $\langle Q_1, \ldots, Q_L \rangle$ of E_i and E_j, respectively - in the sense of Definition 3 - with the properties that (a) the number of preorders decomposing E_j (the more decisive preorder) cannot exceed the number of preorders decomposing E_i (the less decisive preorder), (b) the elements Q_h and P_h of the decompositions of E_j and E_i can be arranged so that either Q_h is an order extension of P_h up to relabelling of vertices, or both Q_h and P_h are complete preorders. Observe finally that the preorder decompositions of (iii) can readily be derived via repeated application of the argument that is developed in the proof of Lemma 1 (see the appendix for further details).

Specifically, observe that in Figure 7 preorder E_j is decomposed into $Q_1 = E_j \cup \{a \succ d\}$, $Q_2 = Trcl(E_j \cup \{a \prec d\})$, and $Q_3 = Trcl(E_j \cup \{a \sim d\})$ and preorder E_i is decomposed into $P_1 = E_i \cup \{a \succ b\}$, $P_2 = Trcl(E_i \cup \{a \prec b\})$, and $P_3 = Trcl(E_i \cup \{a \sim b\})$. Notice that Q_1 and P_1 are order extensions

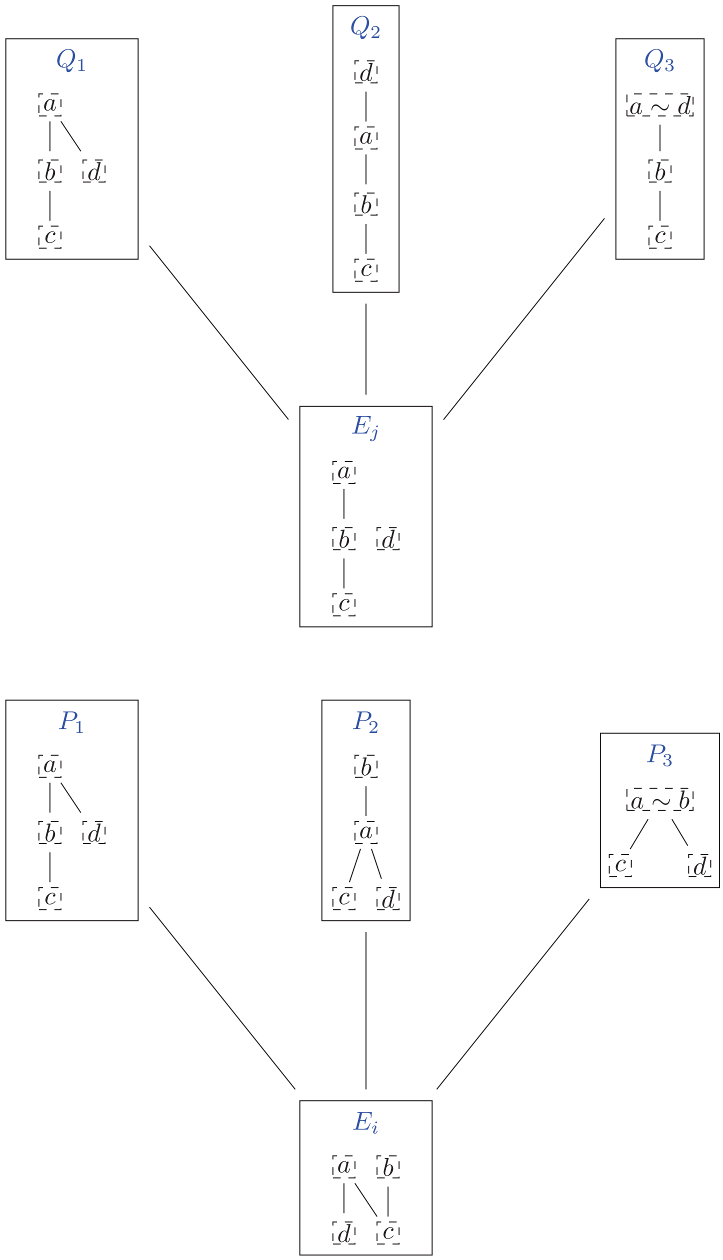

Figure 7: Illustration of Proposition 3

of one another (i.e., they are identical preorders), Q_2 is an order extension of P_2 up to relabelling, and Q_3 is an order extension of P_3 up to relabelling. Thus, by Proposition 3, we conclude that $E_j \succ_{d^{up}} E_i$. In fact, it is readily verified that $|\mathcal{C}(E_j)| = 7$ and $|\mathcal{C}(E_i)| = 11$.

5 Illustrative examples from choice data

To illustrate the usefulness of studying the phenomenon of indecisiveness within the framework of the set of preorder extensions and the decisiveness relations introduced in Definition 2, we consider a number of preorders defined on six groups of goods, from the experimental study by Gerasimou (2021). Subjects in the laboratory were presented with different menus, comprising a number of gift cards (up to six such cards), where each card was worth £10. The cards pertained to two supermarket brands, two coffee-shop chains, one bookstore and a card covering a choice of restaurants. Accordingly, we write the resulting choice set as $X = \{\kappa_a, \kappa_b, \kappa_c, \kappa_d, \kappa_f, \kappa_g\}$, where each κ_i denotes a gift card. Subjects were allowed to express indifference, indecisiveness or a strict preference over the elements in each menu. The subject's preference relation was then constructed by Gerasimou (2021) as the closest preorder to the choices expressed by the respondent.

We first begin with a corollary that explores the logical relation between the above four decisiveness orders.

Corollary 2 (Relation Between the Four Decisiveness Orders)**.** In the set of preorder extensions $(\uparrow E_0, \succeq_{d^{ext}})$, the following implications hold*:*

1. $E_i \succeq_{d^{ext}} E_j \Longrightarrow E_i \succeq_{d^{comp}} E_j \Longrightarrow E_i \succeq_{d^{card-comp}} E_j$,
2. $E_i \succeq_{d^{ext}} E_j \Longrightarrow E_i \succeq_{d^{up}} E_j$

Recall that each of $\succeq_{d^{ext}}$ and $\succeq_{d^{comp}}$ are incomplete relations, while $\succeq_{d^{card-comp}}$ and $\succeq_{d^{up}}$ being metrics, are complete decisiveness relations. From Corollary 2, it follows that the decisiveness metric $\succeq_{d^{card-comp}}$ is an order-preserving function for each of the incomplete relations $\succeq_{d^{ext}}$ and $\succeq_{d^{comp}}$.[13] Likewise, it is the case that the decisiveness metric $\succeq_{d^{up}}$ is an order-preserving function for $\succeq_{d^{ext}}$. It is therefore the case that two examples of order-extensions of

[13]Let $(S, \succeq, \succ)$ denote a preorder. A function $f : S \to \mathbb{R}$ is order-preserving on $(S, \succeq, \succ)$ if for all $a, b \in S$, $a \succeq b$ implies $f(a) \geq f(b)$.

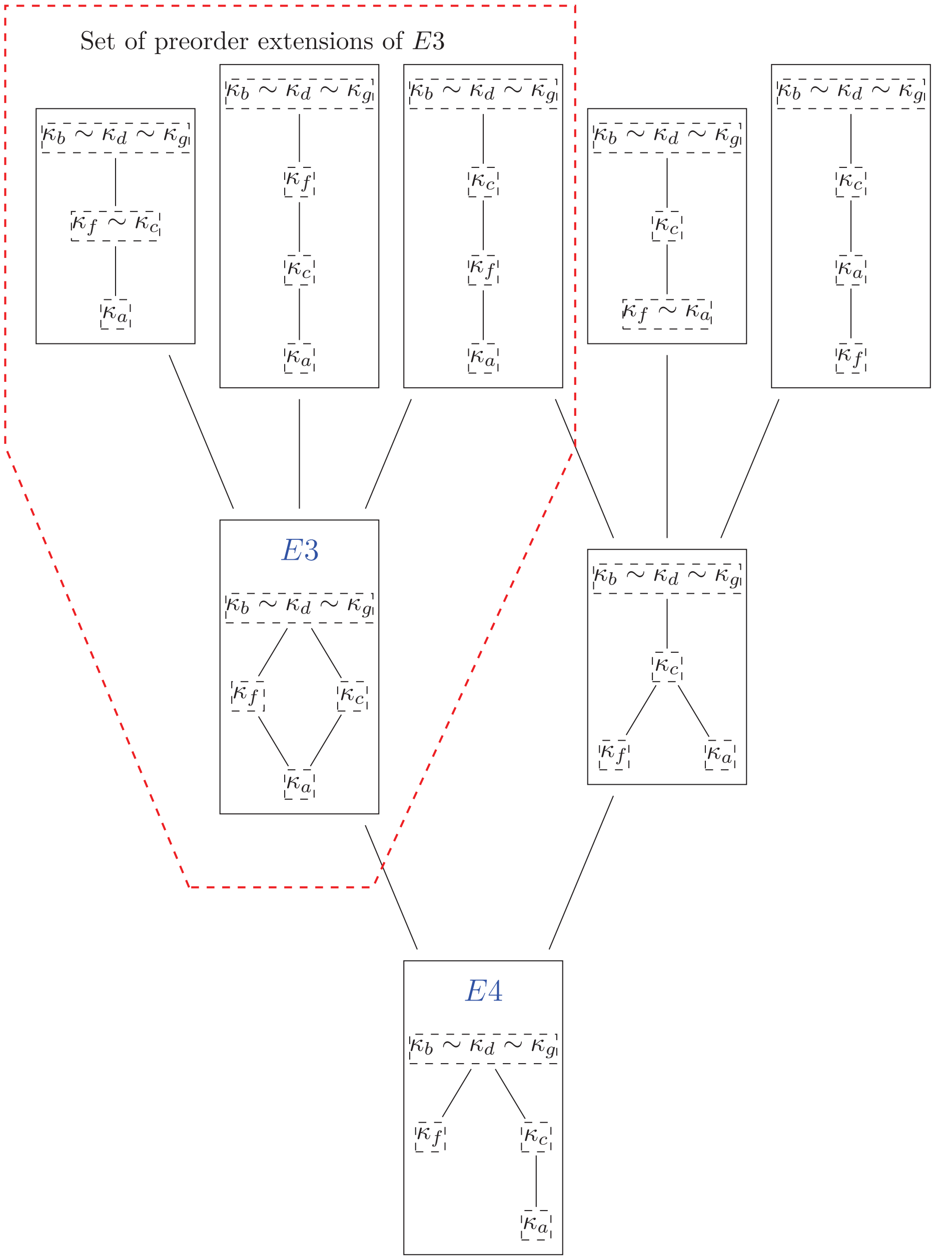

Figure 8: The set of of preorder extensions of E_4

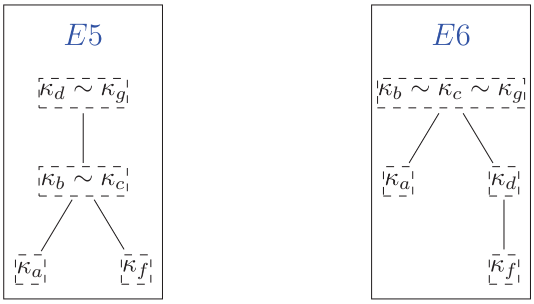

Figure 9: Hasse diagrams of preorders E_5 and E_6. The Hasse diagram of E_6 is identical to the Hasse diagram of E_4 (in figure 8) up to a relabeling of elements.

$\succeq_{d^{ext}}$ that are complete, are given by either of the two decisiveness metrics. With the help of Example 1 and a number of preorders obtained from the experimental choice study of Gerasimou (2021), we shall establish below that no further logical implications arise between the various decisiveness orders of Corollary 2.

First, we consider the preorders E_3 and E_4 of Figure 8.[14] In this particular instance, the preorder E_3 is an extension of E_4 in the sense of Definition 1. For this reason, the set of preorder extensions of E_3 is a subset $\uparrow E_4$. Accordingly, from Corollary 2, E_3 is more decisive than E_4 according to the four decisiveness orders of Definition 2.

E_3 *vs* E_4	E_5 *vs* E_6	E_4 *vs* E_6	E_1 *vs* E_2
$E_3 \succ_{d^{ext}} E_4$	$E_5 \,\|\|_{d^{ext}} E_6$	$E_4 \,\|\|_{d^{ext}} E_6$	$E_1 \,\|\|_{d^{ext}} E_2$
$E_3 \succ_{d^{comp}} E_4$	$E_5 \succ_{d^{comp}} E_6$	$E_4 \,\|\|_{d^{comp}} E_6$	$E_2 \succ_{d^{comp}} E_1$
$E_3 \succ_{d^{card-comp}} E_4$	$E_5 \succ_{d^{card-comp}} E_6$	$E_4 \sim_{d^{card-comp}} E_6$	$E_2 \succ_{d^{card-comp}} E_1$
$E_3 \succ_{d^{up}} E_4$	$E_5 \succ_{d^{up}} E_6$	$E_4 \sim_{d^{up}} E_6$	$E_1 \succ_{d^{up}} E_2$

Table 1: The Four Decisiveness Orderings at Work

[14]The preorders E_3 and E_4 are respectively the preference relations pertaining to subjects with identification numbers 1124 and 8801, whose diagrams are depicted on page 54 of Gerasimou (2021).

Next, consider Figure 9 where we depict the preorders E_5 and E_6.[15] Firstly, note that these preference relations are not comparable according to $\succeq_{d^{ext}}$. For example $\kappa_d \succ^5 \kappa_b$ while $\kappa_b \succ^6 \kappa_d$. On the other hand, because $\text{comp}(E_5) = \text{comp}(E_6) \cup \{(\kappa_a, \kappa_d)\}$, it follows that E_5 is more decisive than the latter preorder according the decisiveness ordering $\succeq_{d^{comp}}$. In turn, from Corollary 2, it follows that E_5 is also more decisive than E_6 according to the metric $\succeq_{d^{card-comp}}$. It is readily verified that the set of complete preorder extensions $\mathcal{C}(E_5)$ has 3 elements. On the other hand, from Figure 8, we observe that $\uparrow E_4$ has 5 elements. Because the metric $\succeq_{d^{up}}$ satisfies the Hasse-Invariance Axiom [A4] and the Hasse diagram of E_6 is equal to that of E_4 up to a relabeling of vertices, we deduce therefore that $\mathcal{C}(E_6)$ has also 5 elements. Accordingly, there results that $E_5 \succ_{d^{up}} E_6$.

Next, the comparison of E_4 and E_6 will enable us to illustrate a few more points of interest. From the above argument and Proposition 1, there results that $E_4 \sim_{d^{card-comp}} E_6$. Furthermore, given that E_3 is more decisive than E_4 according to all four decisiveness orders - in particular, $E_3 \succ_{d^{card-comp}} E_4$ and $E_3 \succ_{d^{up}} E_4$ - as a transitive implication, there results therefore that $E_3 \succ_{d^{card-comp}} E_6$ and $E_3 \succ_{d^{up}} E_6$. On the other hand, because of the relabelling of vertices discussed earlier, there results that E_4 and E_6 are incomparable according to $\succeq_{d^{ext}}$. Because the relabeling involved renders their comparability sets different, there also results that E_4 and E_6 are incomparable according to $\succeq_{d^{comp}}$.

We summarize the above discussion with the help of Table 1. We also append in the last column of the table the comparisons of the preorders E_1 and E_2 of Example 1, using the four decisiveness relations. In this last column, we observe that $E_2 \succ_{d^{card-comp}} E_1$, while $E_1 \succ_{d^{up}} E_2$. It is useful to invoke the interplay between the axioms to show why E_1 is less decisive to E_2 according to $\succeq_{d^{card-comp}}$. Let E' denote the preference relation where $a \succ b \succ c$ and where d is incomparable to each of a, b, c. It follows that E_1 and E' have identical comparability graphs [A3], and from Proposition 1, that $E_1 \sim_{d^{card-comp}} E'$. However, it is clear that E_2 is an order extension of E', and, therefore, by transitivity [A1] and order extension [A2], we have that $E_2 \succ_{d^{card-comp}} E_1$.

On the other hand, it is immediate from Lemma 1 that E_1 has three

[15] These preorders are respectively the preference relations pertaining to subjects with identification numbers 2882 and 5064, whose Hasse diagrams are depicted on page 53 of Gerasimou (2021).

complete preorder extensions: these are the transitive closure of respectively $E_1 \cup \{a \succ d\}$, $E_1 \cup \{a \sim d\}$, and $E_1 \cup \{a \prec d\}$ (see Figure 2). Furthermore, to obtain the five complete preorder extensions of E_2, consider the incomparable pair (b, d). E_2 has an incomplete preorder extension E'' such that $b \succ d$ and two complete preorder extensions - one where $d \succ b$ and one where $b \sim d$. For the incomplete preorder, a further application of Lemma 1 enables us to find three complete preorders that decompose E''. The additivity axiom [B1], therefore, gives the required five complete preorder extensions of E_2. Because E_1 has three complete preorder extensions, if follows that $E_1 \succ_{d^{up}} E_2$.

Because the first two rows of the table involve incomplete decisiveness relations, there invariably results incomparabilities between the various decision-makers. By comparing the first and second rows, it comes out clearly that $\succeq_{d^{comp}}$ is a more complete relation than $\succeq_{d^{ext}}$. Likewise, a comparison of the third row with the first two rows provides an illustration of the order-preserving property of the decisiveness metric $\succeq_{d^{card-comp}}$. Similarly, the order-preserving property of the metric $\succeq_{d^{up}}$ is illustrated by comparing the first and the fourth row of the table.

By inspecting the second and fourth columns of the table we observe that $E_5 \succ_{d^{comp}} E_6$ and $E_5 \succ_{d^{up}} E_6$, while $E_2 \succ_{d^{comp}} E_1$ but $E_1 \succ_{d^{up}} E_2$. From these observations, we may therefore deduce that $\succeq_{d^{comp}}$ and $\succeq_{d^{up}}$ are logically independent decisiveness relations. Furthermore, because from Corollary 2 $\succeq_{d^{comp}}$ implies $\succeq_{d^{card-comp}}$, we may therefore note that $\succeq_{d^{card-comp}}$ and $\succeq_{d^{up}}$ are also logically independent decisiveness relations. We may therefore conclude, on the basis of Table 1 and the above discussion, that there exist no logical implications between the decisiveness orders of Definition 2 - other than those stated in Corollary 2.

6 Concluding comments

We conclude by discussing some of the relative merits of the four decisiveness orders vis-à-vis one another. As a matter of choosing between the various decisiveness orders in the context of experimental and empirical investigations, one guiding principle could consist in examining the nature of the choice set under consideration. Typically, when the choice set involves a range of horizontally differentiated alternatives, there may result a high level of heterogeneity in the responses of decision-makers. As such, on practical grounds, there may be good reasons for working with either of the decisiveness metrics over the incomplete orderings $\succeq_{d^{ext}}$ and $\succeq_{d^{comp}}$. On the other hand, with vertically differentiated alternatives, the incomplete orderings may enable the researcher to compare a larger number of subjects in terms of the indecisiveness they exhibit.

This brings us to our next point, namely the relative advantages incomplete decisiveness relations present over metrics. As is certainly the case in all fields of measurement in the social sciences, there is a tradeoff between enabling the investigator to detect finer differences in decisiveness, and between being able to determine whether one decision-maker is more decisive than another. Namely, the incomplete relations would typically identify a wider range of patterns of decisiveness and nonetheless would provide a less complete ordering than the metric approaches that produce a complete ranking of decision-makers. We therefore suggest that the researcher decides which of a decisiveness order or metric is more suited in the context of their investigation.

One final consideration is of a computational nature. In the context of complex decision environments involving large numbers of alternatives, a metric such as $\succeq_{d^{card-comp}}$ is easily computed in empirical investigations, by simply counting all comparable pairs of elements by the relation. The computation of the decisiveness metric $\succeq_{d^{up}}$ is somewhat less direct. Nonetheless, this task is computationally feasible via repeated applications of the preorder decomposition proposed in Lemma 1 of this paper.

Appendix

Independence of the Axioms ($d^{card-comp}$ metric)

We prove that the axioms that characterise the $\succeq_{d^{card-comp}}$ metric are independent via five examples. To do so, we will - in each example - specify the ground set X, and on X explicitly define some preorders in $\uparrow E_0$ by omitting (for brevity) to spell out the reflexive parts of the relevant binary relations.

- **A2, A3, A4, and A5 do not imply A1**. Let $X = \{x, y, z\}$. Let $C_1 = \{x \sim y\}$, $C_2 = \{x \succ y\}$, and $C_3 = \{y \succ z\}$ be preorders in

$\uparrow E_0$. Suppose that $\succeq_d$ coincides with $\succeq_{d^{card-comp}}$, except that C_1 is $\succeq_d$-incomparable to C_3. Observe that A2, A3, and A4 are satisfied. In addition, A5 vacuously holds. However, notice that A1 is violated, as $C_1 \sim_d C_2$ (by A3) and $C_2 \sim_d C_3$ (by A4), and yet C_1 and C_3 are $\succeq_d$-incomparable.

- **A1, A3, A4, and A5 do not imply A2**. Let $X = \{x, y\}$. Let $C_1 = \{x \succ y\}$, $C_2 = \{x \prec y\}$, $C_3 = \{x \sim y\}$, and $E_0 = \{\}$ be preorders in $\uparrow E_0$. Suppose that $\succeq_d$ coincides with $\succeq_{d^{card-comp}}$, except that E_0 is $\succeq_d$-incomparable to C_1, C_2, and C_3. The resulting relation $\succeq_d$ satisfies A1, A3, and A4. Furthermore, it vacuously satisfies A5. However, observe that A2 is violated, as C_1 is a preorder extension of E_0, and yet C_1 and E_0 are incomparable.

- **A1, A2, A4, and A5 do not imply A3**. Let $X = \{x, y\}$. Reconsider the preorders defined in the previous bullet point. Suppose that $\succeq_d$ coincides with $\succeq_{d^{card-comp}}$, except that C_3 is $\succeq_d$-incomparable to C_1 and C_2. Notice that A1, A2, and A4 are satisfied. Moreover, A5 vacuously holds. However, observe that A3 is violated, because C_3 and C_1 have the same comparability graph, and yet they are incomparable.

- **A1, A2, A3, and A5 do not imply A4**. Let $X = \{x, y, z\}$. Let $C_1 = \{x \succ y, x \succ z\}$, $C_2 = \{y \succ x, z \succ x\}$, $C_3 = \{y \succ x, y \succ z\}$, $C_4 = \{x \succ y, z \succ y\}$, $C_5 = \{z \succ x, z \succ y\}$, and $C_6 = \{x \succ z, y \succ z\}$ be preorders in $\uparrow E_0$. Suppose that $\succeq_d$ is identical to $\succeq_{d^{card-comp}}$, except that (i) C_1 and C_2 are incomparable to C_3, C_4, C_5, C_6, (ii) C_3 and C_4 are incomparable to C_1, C_2, C_5, C_6, and (iii) C_5 and C_6 are incomparable to C_1, C_2, C_3, C_4. Observe that A1, A2, and A3 hold. Furthermore, A5 vacuously holds. However, notice that A4 is violated, because A4 requires the inverted V-shaped posets to be equally decisive, and yet C_1, C_3, and C_5 are incomparable. Similarly, A4 requires the V-shaped posets to be equally decisive, and yet C_2, C_4, and C_6 are incomparable.

- **A1, A2, A3, and A4 do not imply A5**. Let $X = \{x, y, z, w\}$. Let $C_1 = \{x \succ w\}$, $C_2 = \{x \succ z\}$, $C_3 = \{x \succ w, y \succ w\}$, $C_4 = \{x \succ z, y \succ w\}$ be preorders in $\uparrow E_0$. Suppose that $\succeq_d$ is identical to $\succeq_{d^{card-comp}}$, except that C_3 is incomparable to C_4. Observe that $\succeq_d$ satisfies A1, A2, A3, and A4. However, noticed that A5 is violated, because (i) $C_1 \sim_d C_2$ (by A4), (ii) $y \succ w$ is a passive pair for both C_1 and C_2, (iii)

$C_3 = C_1 \cup \{y \succ w\}$ and $C_4 = C_2 \cup \{y \succ w\}$, and yet C_3 is incomparable to C_4.

Independence of the Axioms (d^{up} metric)

Consider a two-element choice set $X = \{x, y\}$. Throughout, we omit (for brevity) to specify the reflexive parts of the preorders on X. Define four preorders on X as follows: $F_1 = \{x \succ y\}$, $F_2 = \{x \prec y\}$, $F_3 = \{x \sim y\}$, and $E_0 = \{\}$. The set of preorder extensions of E_0 is thus $\uparrow E_0 = \{F_1, F_2, F_3, E_0\}$. Let $f : \uparrow E_0 \to \mathbb{N}$ be a function that measures the degree of indecisiveness of a preorder in $\uparrow E_0$. Consider the following three specifications.

- f **satisfies Normalisation [B1], but violates Additivity [B2]**: let $f(F_1) = 1$, $f(F_2) = 1$, $f(F_3) = 1$, and $f(E_0) = 5$.
- f **satisfies Additivity [B2], but violates Normalisation [B1]**: let $f(F_1) = 2$, $f(F_2) = 3$, $f(F_3) = 4$, and $f(E_0) = 9$.

Proof of Proposition 1

Necessity. The necessity part is readily verified, and thus omitted. Below we prove the sufficiency part.

Sufficiency. Let $\succeq_d$ denote some decisiveness relation on the set of preorder extensions. Assume that $\succeq_d$ satisfies axioms A1-A5. We want to show that $E_j \succeq_d E_i$ whenever $|\text{comp}(E_j)| \geq |\text{comp}(E_i)|$. We divide the proof in three steps.

Step 1: Let $Ext(E_0)$ denote the set of poset extensions of an n-element antichain. Then, there exists a surjective function $f : \uparrow E_0 \to Ext(E_0)$ such that, for all $E_i, E_j \in \uparrow E_0$, if $E_j \succeq_d E_i$, then $f(E_j) \succeq_d f(E_i)$.

Proof. Let E_i be a preorder in $\uparrow E_0$ that is not a poset. Denote by $p(E_i)$ a poset constructed as follows: $p(E_i)$ is identical to E_i, except that every non-singleton indifference class in E_i is turned into a chain in $p(E_i)$.

Define a function $f : \uparrow E_0 \to Ext(E_0)$ as follows. For any $E_i \in \uparrow E_0$, let

$$f(E_i) := \begin{cases} E_i & \text{if } E_i \text{ is a poset in } \uparrow E_0 \\ p(E_i) & \text{if } E_i \text{ is a preorder in } \uparrow E_0 \text{ that is not a poset} \end{cases}$$

By construction, the image of f is the set $Ext(E_0)$ of poset extensions. Hence, f is surjective. It remains to show that for all $E_i, E_j \in \uparrow E_0$, if $E_j \succeq_d E_i$, then $f(E_j) \succeq_d f(E_i)$. Assume that $E_j \succeq_d E_i$. We distinguish three cases.

Case (i): assume that both E_i and E_j are posets in $\uparrow E_0$. Since $f(E_i) = E_i$ and $f(E_j) = E_j$, then $f(E_j) = E_j \succeq_d E_i = f(E_i)$, as desired.

Case (ii): suppose that both E_i and E_j are preorders in $\uparrow E_0$ that are not posets. By comparability graph invariance [A3], $E_i \sim_d p(E_i) = f(E_i)$ and $E_j \sim_d p(E_j) = f(E_j)$. Since, by assumption, $E_j \succeq_d E_i$, then $f(E_j) \sim_d E_j \succeq_d E_i \sim_d f(E_i)$, which, by transitivity [A1], implies that $f(E_j) \succeq_d f(E_i)$, which is the desired result.

Case (iii): assume WLOG that E_i is a poset in $\uparrow E_0$ and E_j is a preorder in $\uparrow E_0$ that is not a poset. By similar arguments to those of case (ii), it follows that $f(E_j) \succeq_d f(E_i)$. □

By Step 1, for every preorder $E_i \in \uparrow E_0$ that is not a poset, there exists a poset $p(E_i)$ that is as decisive as preorder E_i. This result enables us to complete the proof by considering the set of poset extensions $Ext(E_0)$, which - as discussed in the main body - is a subset of the set $\uparrow E_0$ of preorder extensions. From Brualdi, Jung, and Trotter (1994), (a) $(Ext(E_0), \succeq_{d^{ext}})$ is a ranked poset and (b) every poset E_i in $(Ext(E_0), \succeq_{d^{ext}})$ has at least two lower covers.

Step 2: For any $E_i, E_j \in Ext(E_0)$ such that $|\text{comp}(E_i)| = |\text{comp}(E_j)|$, there holds $E_i \sim_d E_j$.

Proof. Let $E_i, E_j \in Ext(E_0)$ be such that $|\text{comp}(E_i)| = |\text{comp}(E_j)|$. We prove that $E_i \sim_d E_j$ by induction on the cardinality of $\text{comp}(E_i)$ and $\text{comp}(E_j)$.

`Base case`: assume that $|\text{comp}(E_i)| = |\text{comp}(E_j)| = 1$. Notice that all posets with one comparability have the same Hasse diagram that is given by a 2-element chain and an $(n-2)$-element antichain. Hence, by Hasse diagram invariance [A4], $E_i \sim_d E_j$.

`Inductive hypothesis`: suppose that for any $E_i, E_j \in Ext(E_0)$ such that $|\text{comp}(E_i)| = |\text{comp}(E_j)| = c$, there holds $E_i \sim_d E_j$ for all $1 \leq c < \frac{n(n-1)}{2}$.

Consider any $E_i, E_j \in Ext(E_0)$ such that $|\text{comp}(E_i)| = |\text{comp}(E_j)| = c+1$. We want to show that $E_i \sim_d E_j$.

Let $E_k \in Ext(E_0)$ be such that $|\text{comp}(E_k)| = c$ and $E_i \succ_{d^{ext}} E_k$. By Brualdi, Jung, and Trotter (1994), every poset E_i has a lower cover in $(Ext(E_0), \preceq_{d^{ext}})$. Hence, E_k exists. Moreover, since $|\text{comp}(E_i)| = c+1$, $|\text{comp}(E_k)| = c$, and $E_i \succ_{d^{ext}} E_k$, then there exists a pair of distinct alternatives $(a,b) \in X \times X$ such that $E_i = E_k \cup \{a \succ b\}$. We distinguish two cases.

Case (a): suppose that $(a \succ b) \in E_j$ and E_j has a lower cover E_l in $(Ext(E_0), \succeq_{d^{ext}})$ such that $E_j = E_l \cup \{(a \succ b)\}$. Observe that $|\text{comp}(E_k)| = |\text{comp}(E_l)| = c$. Hence, by the inductive hypothesis, $E_k \sim_d E_l$. Since - by assumption - $E_i = E_k \cup \{(a \succ b)\}$ and $E_j = E_l \cup \{(a \succ b)\}$, then - by independence [A5] - it follows that $E_i \sim_d E_j$, which is the desired result.

Case(b): assume that either (i) $(a \succ b) \notin E_j$ or (ii) $(a \succ b) \in E_j$ and E_j does not have a lower cover E_l in $(Ext(E_0), \succeq_{d^{ext}})$ such that $E_j = E_l \cup \{(a \succ b)\}$. Since $E_j \in Ext(E_0)$, then it has a lower cover. Now construct another poset in $Ext(E_0)$ - call it $\hat{E}_j$ - that satisfies the following properties: (1) $\hat{E}_j$ has the same Hasse diagram as E_j, and (2) the alternatives in $\hat{E}_j$ are relabelled in order to make sure that $\hat{E}_j$ has a lower cover - call it $\hat{E}_l$ - such that $\hat{E}_j = \hat{E}_l \cup \{(a \succ b)\}$. By Hasse diagram invariance [A4], $\hat{E}_j \sim_d E_j$. By the same arguments as those used in case (a), $E_k \sim \hat{E}_l$ implies that $E_i = E_k \cup \{a \succ b\} \sim_d \hat{E}_l \cup \{(a \succ b)\} = \hat{E}_j$. Hence, $E_i \sim_d \hat{E}_j$. Since, by Hasse diagram invariance [A4], $\hat{E}_j \sim_d E_j$, then - by transitivity [A1] - $E_i \sim_d E_j$, as desired. □

Step 3: For any $E_i, E_j \in Ext(E_0)$ such that $|\text{comp}(E_i)| < |\text{comp}(E_j)|$, there holds $E_j \succ_d E_i$.

Proof. By Step 2, for any $E_i, E_j \in Ext(E_0)$ such that $|\text{comp}(E_i)| = |\text{comp}(E_j)|$, there holds $E_i \sim_d E_j$. Consider now any two posets $E_i, E_j \in Ext(E_0)$ such that $|\text{comp}(E_i)| < |\text{comp}(E_j)|$. We want to show that $E_j \succ_d E_i$. We distinguish two cases.

Case (a): suppose that E_j is an order extension of E_i. Then, by order-extension [A2], it immediately follows that $E_j \succ_d E_i$.

Case (b): suppose that E_j is not an order extension of E_i. By Brualdi, Jung, and Trotter (1994), $(Ext(E_0), \succeq_{d^{ext}})$ is a ranked poset. Hence, there exists a maximal chain $\mathcal{M}$ in $(Ext(E_0), \succeq_{d^{ext}})$ such that $E_i, E_m \in \mathcal{M}$, $E_m \succ_{d^{ext}} E_i$, and $|\text{comp}(E_m)| = |\text{comp}(E_j)|$. By order-extension [A2], $E_m \succ_d E_i$, and, by Step 2, $E_m \sim_d E_j$. Therefore, by transitivity [A1], it follows that $E_j \succ_d E_i$, which is the desired result.

□

By Step 2 and Step 3, $\succeq_d = \succeq_{d^{card-comp}}$. This concludes the sufficiency part of the proof.

Proof of Lemma 1

Let $E \in \uparrow E_0$ be an incomplete preorder. Since E is not complete, there exists a pair $(x, y) \in X \times X$ such that x and y are not comparable in E. Given a preorder $S = (X, \succeq_S)$ and a pair of alternatives $a, b \in X$ such that a and b are S-incomparable, let $Trcl(S \cup \{a \succ_S b\})$ denote the transitive closure of $S \cup \{a \succ_S b\}$.

Define three distinct preorders P, Q, R as follows.

$$P := Trcl(E \cup \{x \succ y\})$$
$$Q := Trcl(E \cup \{y \succ x\})$$
$$R := Trcl(E \cup \{x \sim y\})$$

We first verify that $P \cap Q \cap R = E$. There are three cases to consider.

Case (1): $x \succ y$ and $y \succ x$ are passive pairs for E. This necessarily implies that, for all $z \in X \setminus \{x, y\}$, $x \succ z$ if and only if $y \succ z$; likewise, $z \succ x$ if and only if $z \succ y$. Therefore, case (1) implies that $x \sim y$ is a passive pair for E. Consider first $x \succ y$. Observe that the transitive closure of $E \cup \{x \succ y\}$ is equal to $E \cup \{x \succ y\}$ itself, because $x \succ y$ is a passive pair for E. Likewise, it is the case that $Trcl(E \cup \{x \prec y\}) = E \cup \{x \prec y\}$ and $Trcl(E \cup \{x \sim y\}) = E \cup \{x \sim y\}$. As a result, it follows that the intersection of P, Q, and R equals E.

Case (2): $x \succ y$ is a passive pair for E and $y \succ x$ is not a passive pair for E. Observe that this necessarily implies that $x \sim y$ is *not* a passive pair for E.[16] As argued in the case above, the transitive closure of $E \cup \{x \succ y\}$

[16]Observe WLOG that the case where $x \succ y$ is *not* a passive pair for E and $y \succ x$ is a passive pair for E is treated analogously.

is equal to $E \cup \{x \succ y\}$, because $x \succ y$ is a passive pair for E. On the other hand, $Trcl(E \cup \{y \succ x\}) = E \cup \{y \succ x\} \cup T$, where T is the resulting set of ordered pairs from $X \times X$ that are added in the transitive closure. Likewise, the same set T characterises the transitive closure of $E \cup \{x \sim y\}$: that is, we have that $Trcl(E \cup \{y \sim x\}) = E \cup \{y \sim x\} \cup T$. Therefore,

$$\begin{aligned} P \cap Q \cap R &= [Trcl(E \cup \{x \succ y\})] \cap [Trcl(E \cup \{y \succ x\})] \cap [Trcl(E \cup \{x \sim y\})] \\ &= [E \cup \{x \succ y\}] \cap [E \cup \{y \succ x\} \cup T] \cap [E \cup \{y \sim x\} \cup T] \end{aligned}$$

As a result, $P \cap Q \cap R = E$, because the pairs of alternatives in T are not comparable in P.

Case (3): $x \succ y$ is not a passive pair for E and $y \succ x$ is not a passive pair for E. Observe that this also implies that $x \sim y$ is *not* a passive pair for E. For sets $T_1, T_2 \subset X \times X$, from the argument developed in case (2) above, we have:

$$\begin{aligned} Trcl(E \cup \{x \succ y\}) &= E \cup \{x \succ y\} \cup T_1 \\ Trcl(E \cup \{y \succ x\}) &= E \cup \{y \succ x\} \cup T_2 \\ Trcl(E \cup \{y \sim x\}) &= E \cup \{x \sim y\} \cup T_1 \cup T_2 \end{aligned}$$

where it must be the case that $T_1 \cap T_2 = \emptyset$. As a result, it follows again that $P \cap Q \cap R = E$.

Observe that cases (1), (2), and (3) above exhaust all possibilities and are mutually exclusive. This therefore establishes that $P \cap Q \cap R = E$.

We are now ready to prove that $\langle \mathcal{C}(P), \mathcal{C}(Q), \mathcal{C}(R) \rangle$ forms a partition of $\mathcal{C}(E)$. It is readily verified that, by construction, $\mathcal{C}(P) \cap \mathcal{C}(Q) = \mathcal{C}(P) \cap \mathcal{C}(R) = \mathcal{C}(Q) \cap \mathcal{C}(R) = \emptyset$.

Hence, it remains to show that there is no complete preorder $F \in \uparrow E_0$ such that $F \notin \mathcal{C}(P) \cup \mathcal{C}(Q) \cup \mathcal{C}(R)$ and yet $F \in \mathcal{C}(E)$. Suppose not.

Since F is a complete preorder, then it must be able to order (x, y). Suppose WLOG that $x \succ_F y$.[17] Since F and P order (x, y) in the same way and $F \in \mathcal{C}(E) \setminus \mathcal{C}(P)$, then there must be some other pair of alternatives (z, w) that F and P order differently. We distinguish two cases.

[17] The cases where $x \sim_F y$ and $y \succ_F x$ are similarly handled. See in particular the first part of the proof.

Case (1): $x \succ y$ is a passive pair for E. That is, $P = Trcl(E \cup \{x \succ y\}) = E \cup \{x \succ y\}$. Since $F \in \mathcal{C}(E)$ and $x \succ_F y$, then it means that $a \succeq_P b \Rightarrow a \succeq_F b$, implying that F is a preorder extension of P. However, this contradicts the fact that there is a pair of alternatives (z, w) that F and P order differently.

Case (2): $x \succ y$ is not a passive pair for E. Therefore, $E \cup \{x \succ y\} \subset Trcl(E \cup \{x \succ y\}) = P$. This means that a necessary condition for resolving the incomparability between x and y by having $x \succ y$ is to resolve some other incomparabilities. Specifically, let $T := Trcl(E \cup \{x \succ y\}) \setminus (E \cup \{x \succ y\})$ denote the set of extra ordered pairs that result from taking the transitive closure of $E \cup \{x \succ y\}$. Hence, for any $(v \succeq w) \in T$ and any order extension E' of E, it follows that, if $x \succ_{E'} y$, then $v \succeq_{E'} w$. Therefore, since by assumption $x \succ_F y$ and F is an order extension of E, then it follows that $v \succeq_F w$. Recall that $P = E \cup \{x \succ y\} \cup T$. But then this implies that there is no pair of alternatives (z, w) that F and P order differently. The latter produces the desired contradiction.

Hence, there is no complete preorder $F \in \uparrow E_0$ such that $F \notin \mathcal{C}(P) \cup \mathcal{C}(Q) \cup \mathcal{C}(R)$ and yet $F \in \mathcal{C}(E)$. Thus, $\langle \mathcal{C}(P), \mathcal{C}(Q), \mathcal{C}(R) \rangle$ forms a covering of $\mathcal{C}(E)$. Observe that $\langle \mathcal{C}(P), \mathcal{C}(Q), \mathcal{C}(R) \rangle$ is also a partition of $\mathcal{C}(E)$, as from the choices of P, Q, and R, there cannot be a complete preorder in the covering that is not an extension of $\mathcal{C}(E)$. This concludes the proof of the lemma.

Proof of Proposition 2

Necessity. It is readily verified that, if f counts the number of complete preorder extensions, then [B1] and [B2] hold.

Sufficiency. Assume that f satisfies Additivity [B1] and Normalisation [B2]. Let n^* denote the number of complete preorder extensions of an n-element antichain. Define a finite set

$$\mathcal{S}(\uparrow E_0) := \{s \in \mathbb{N} : \text{there is a preorder } E \in \uparrow E_0 \text{ such that } s = |\mathcal{C}(E)|\}$$

containing all the natural numbers s that have the property that there is a preorder E such that the number of complete preorder extensions of E is exactly equal to s. We observe that $\min \mathcal{S}(\uparrow E_0) = 1$ and $\max \mathcal{S}(\uparrow E_0) = n^*$. For convenience, denote the elements of $\mathcal{S}(\uparrow E_0)$ by $\{s_1, \ldots, s_i, \ldots, s_M\}$, where $s_1 = 1$, $s_M = n^*$, and $s_i < s_{i+1}$ for all $i \in \{1, \ldots, M-1\}$, .

Let $E \in\uparrow E_0$ be a preorder, and let $\mathcal{C}(E)$ denote the set of complete preorder extensions of E. We want to show that $f(E) = |\mathcal{C}(E)|$. We argue by induction on $|\mathcal{C}(E)| \in \{s_1, \ldots, s_i, \ldots, s_M\}$.

`Base Step`: assume that $|\mathcal{C}(E)| = s_1 = 1$. Then, E is a complete preorder, and has one complete preorder extension, i.e., itself. By Normalisation [B2], $f(E) = 1$. Thus, $f(E) = |\mathcal{C}(E)|$.

`Inductive Step`: assume that $f(E) = |\mathcal{C}(E)| = s_i$ for all $1 \leq i < M$.

Consider a preorder $E \in\uparrow E_0$ such that $|\mathcal{C}(E)| = s_{i+1}$. We want to show that $f(E) = |\mathcal{C}(E)| = s_{i+1}$.

Let $E \in\uparrow E_0$ be such a preorder. Since $|\mathcal{C}(E)| = s_{i+1} > 1$, then there exists a pair of alternatives (x, y) such that x and y are not comparable at E. Let $P := Trcl(E \cup \{x \succ y\})$, $Q := Trcl(E \cup \{x \prec y\})$, and $R := Trcl(E \cup \{x \sim y\})$. By Lemma 1, $\langle \mathcal{C}(P), \mathcal{C}(Q), \mathcal{C}(R) \rangle$ forms a partition of $\mathcal{C}(E)$. By Additivity [B1], $f(E) = f(P) + f(Q) + f(R)$. Since, by construction, P, Q, R are preorder extensions of E, then $\max\{|\mathcal{C}(P)|, |\mathcal{C}(Q)|, |\mathcal{C}(R)|\} \leq s_i < s_{i+1}$. Therefore, by the inductive step, $f(P) = |\mathcal{C}(P)|$, $f(Q) = |\mathcal{C}(Q)|$, and $f(R) = |\mathcal{C}(R)|$. Hence, $f(E) = |\mathcal{C}(P)| + |\mathcal{C}(Q)| + |\mathcal{C}(R)|$.

On the other hand, suppose, by contradiction, that $|\mathcal{C}(E)| \neq |\mathcal{C}(P)| + |\mathcal{C}(Q)| + |\mathcal{C}(R)|$. Recall that, by Lemma 1, $\langle \mathcal{C}(P), \mathcal{C}(Q), \mathcal{C}(R) \rangle$ forms a partition of $\mathcal{C}(E)$. Hence, it cannot be that $|\mathcal{C}(E)| > |\mathcal{C}(P)| + |\mathcal{C}(Q)| + |\mathcal{C}(R)|$ or $|\mathcal{C}(E)| < |\mathcal{C}(P)| + |\mathcal{C}(Q)| + |\mathcal{C}(R)|$. Therefore, $|\mathcal{C}(E)| = |\mathcal{C}(P)| + |\mathcal{C}(Q)| + |\mathcal{C}(R)| = f(E)$, which is the desired result.

Proof of Proposition 3

By Proposition 2, (i) $\iff$ (ii). Hence, it remains to show that (ii) $\iff$ (iii). We begin by showing that (ii) $\Rightarrow$ (iii). Assume that (ii) holds. Since (i) $\iff$ (ii), then $|\mathcal{C}(E_j)| = f(E_j) \leq f(E_i) = |\mathcal{C}(E_i)|$. Let $K := f(E_i)$ and $L := f(E_j)$. Let $\mathcal{C}(E_j) = \left\{C_1^{E_j}, \ldots, C_L^{E_j}\right\}$ and $\mathcal{C}(E_i) = \left\{C_1^{E_i}, \ldots, C_K^{E_i}\right\}$ be the set of complete preorder extensions of E_j and E_i, respectively. Define $Q_h := C_h^{E_j}$ for all $h \in \{1, \ldots, L\}$. Observe that, from Szpilrajn (1930)'s theorem, $\langle Q_1, \ldots, Q_L \rangle$ so defined is a preorder decomposition of E_j. Similarly, define $P_h := C_h^{E_i}$ for all $h \in \{1, \ldots, K\}$ and observe that $\langle P_1, \ldots, P_K \rangle$ is a preorder decomposition of E_i. It remains to show that (iii.a) and (iii.b) hold. Since

(ii) holds, it follows - by construction - that $L = f(E_j) \leq f(E_i) = K$. Hence, (iii.a) holds. On the other hand, both P_h and Q_h are complete preorders. Hence, (iii.a) holds as well. Therefore, we have shown that (ii) $\Rightarrow$ (iii).

In the other direction, we now show that (iii) $\Rightarrow$ (ii). Assume that (iii) holds. Let $\langle P_1, \ldots, P_K \rangle$ and $\langle Q_1, \ldots, Q_L \rangle$ be preorder decompositions of E_i and E_j, respectively, where P_h and Q_h need not be complete preorders. By definition of preorder decomposition, we have:

$$|\mathcal{C}(E_j)| = |\mathcal{C}(Q_1)| + \cdots + |\mathcal{C}(Q_L)| \tag{5}$$

$$|\mathcal{C}(E_i)| = |\mathcal{C}(P_1)| + \cdots + |\mathcal{C}(P_K)| \tag{6}$$

Since (iii) holds and $K \geq L$, we have that - for each $h \in \{1, \ldots, L\}$ - either $Q_h \succeq_{d^{ext}} P_h$, up to a relabelling of the vertices, or P_h and Q_h are both complete preorders. This implies that $|\mathcal{C}(Q_h)| \leq |\mathcal{C}(P_h)|$ for each $h = \{1, \ldots, L\}$. In the light of equation 5, equation 6, and the above inequalities, it follows that $|\mathcal{C}(E_j)| \leq |\mathcal{C}(E_i)|$. That is, $E_j \succeq_{d^{up}} E_i$. Hence, we have shown that (iii) $\Rightarrow$ (i). Since (i) $\Longleftrightarrow$ (ii), then we have also shown that (iii) $\Rightarrow$ (ii), which is the desired result.

References

Anderson, C. J. (2003): "The Psychology of Doing Nothing: Forms of Decision Avoidance Result From Reason and Emotion," *Psychological Bulletin*, 129(1), 139–167.

Brualdi, R. A., H. C. Jung, and W. T. Trotter (1994): "On the Poset of All Posets on n Elements," *Discrete Applied Mathematics*, 50, 111–123.

Caspard, N., B. Leclerc, and B. Monjardet (2012): *Finite Ordered Sets*. Cambridge University Press.

Cettolin, E., and A. Riedl (2019): "Revealed preferences under uncertainty: Incomplete preferences and preferences for randomization," *Journal of Economic Theory*, 181, 547–585.

Chambers, C. P., and F. Echenique (2016): *Revealed Preference Theory.* Econometric Society Monograph. Cambridge University Press.

Costa-Gomes, M., C. Cueva, G. Gerasimou, and M. Tejiščák (2022): "Choice, Deferral and Consistency," *Quantitative Economics*, 13(3), 1297–1318.

Danan, E., and A. Ziegelmeyer (2006): "Are Preferences Complete? An Experimental Measurement of Indecisiveness Under Risk," Working Paper.

Dubra, J., F. Maccheroni, and E. A. Ok (2004): "Expected Utility Theory without the Completeness Axiom," *Journal of Economic Theory*, 115, 118–133.

Eliaz, K., and E. A. Ok (2006): "Indifference or Indecisiveness? Choice-Theoretic Foundations of Incomplete Preferences," *Games and Economic Behavior*, 56, 61–86.

Fudenberg, D., and D. K. Levine (2006): "A Dual-Self Model of Impulse Control," *American Economic Review*, 96(5), 1449–1476.

Gerasimou, G. (2018): "Indecisiveness, Undesirability, and Overload Revealed Through Rational Choice Deferral," *Economic Journal*, 128, 2450–2479.

——— (2021): "Model-Rich Approaches to Eliciting Possibly Weak or Incomplete Preferences: Evidence from a Multi-Valued Choice Experiment," arXiv:submit/4097132[econ.TH]29DEC2021.

Gorno, L. (2018): "The Structure of Incomplete Preferences," *Economic Theory*, 66, 159–185.

Ok, E. A. (2002): "Utility Representation of an Incomplete Preference Relation," *Journal of Economic Theory*, 104, 429–449.

Qiu, J., and Q. Ong (2017): "Indifference or Indecisiveness: a Strict Discrimination," MPRA Working Paper No. 81440.

Richter, M. K. (1966): "Revealed Preference Theory," *Econometrica*, 34(3), 635–645.

Szpilrajn, E. (1930): "Sur l'extension de l'ordre partiel," *Fundamental Mathematicae*, 16(3), 386–389.