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Ordering Distributions on a Finitely Generated Cone

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Abstract

One large class of relations used in the measurement of social welfare and risk consists of relations induced by finitely generated cones. Within this class, we develop a general approach to investigate the ordering of distributions–an approach that does not require the prior derivation of a numerical representation of the order relation. We provide an equivalence between the statement that two distributions x and y are ordered, and (1) the possibility of expressing $x - y$ as a positive combination of a subset of linearly independent vectors from the generators of the cone, (2) the existence of a relation defined on a simplicial cone such that x and y are ordered by this latter relation, and (3) the existence of a generalized inverse G of the matrix whose columns generate the underlying cone, such that the product of G and the vector $x - y$ results in a non-negative vector. The results are illustrated in the context of distributional comparisons on socioeconomic data.

Keywords Measurement of social welfare and inequality, order relations induced by convex cones, Carathéodory's theorem, Farkas lemma, generalized inverses.

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1. Introduction

While in the standard theory of individual choice a preference relation is taken to be transitive and complete, a social welfare ordering is typically not assumed to order every pair of social states. Likewise, in the theory of choice under risk, a preference relation does not order all lotteries. Thus, when ordering distributions in terms of social welfare or risk, the underlying preference relation is often assumed to take the form of a preorder; that is a reflexive, transitive, and possibly incomplete relation. As recently emphasized by Magdalou (2021), many (if not most) incomplete preorders studied in the fields of distributional analysis and choice under risk, arise as order relations induced by finitely generated cones. In applied work involving incomplete preference relations, it is therefore of central importance to derive a numerical implementation criterion, or equivalently a multi-utility representation, for investigating the ordering of a pair of distributions. However, as argued by Ok (2002), the difficulty of finding such a representation is rooted in the incompleteness of the order relation. At an abstract level, the derivation of such a criterion requires the researcher to identify a number of complete order extensions, with the additional requirement that the intersection of these complete relations produces the social welfare, or risk, ordering.

Thus, the purpose of this paper is to develop a new approach to investigate the ordering of distributions defined on finitely generated cones–an approach that does not require the prior derivation of a numerical / multi-utility representation of the order relation. Examples where the approach developed in this paper may be applicable include widely used relations, such as first and second order stochastic dominance, the Lorenz inequality ordering and the Hammond order (Gravel, Magdalou and Moyes, 2021) used to compare distributions defined on ordinal data. The approach developed in this paper is appealing in applied work, as the same set of criteria we develop can be applied within the entire class of order relations defined on finitely generated cones, thus side-stepping the often complex task of finding a numerical representation of each particular relation within the class.

The main result of this paper is a theorem that provides an equivalence between four statements: (*i*) the fact that two distributions x and y are ordered by a relation defined on a finitely generated cone, (*ii*) the possibility of expressing $x - y$ as a positive combination of a subset of linearly independent vectors from the set of generators of the cone, (*iii*) the existence of a *simplicial* relation (a relation defined on a simplicial cone), such that x and y are ordered by this latter relation, and (*iv*) the existence of a generalized inverse G of the matrix whose columns are the generators of the underlying cone, such that the product of G and the vector $x - y$ results in a non-negative vector.

The significance of the result is as follows. Call the relation defined on the finitely generated cone the *mother relation.* The simplicial relation is shown to be coarser than the mother relation, in the sense that the former only enables the researcher to order a subset of the pairs of distributions that are comparable in the mother relation. Nonetheless, the simplicial relation is easily characterized by expressing $x - y$ as a positive combination of a subset of linearly independent vectors from the set of generators of the cone. This set of vectors (also known as a *positive basis*) fully determines the simplicial relation or, equivalently, the associated simplicial cone. Furthermore, the same generators of the simplicial cone provide all the information needed in order to construct the generalized inverse G of the matrix of the generators of the cone of the mother relation. The generalized inverse in statement (*iv*) in turn provides a new empirical criterion, simple to implement, in order to investigate the comparability of a pair of distributions x and y.

As a key step for making this approach appealing in applied work is to be able to find a positive basis, the paper also discusses various methods for obtaining the set of linearly independent vectors that positively span $x - y$. We also note that finding such positive bases is routinely required in linear programming problems, and we adapt to the context discussed in this paper one such algorithm due to Schrijver (1986). Finally, to illustrate the practical relevance of the methodology introduced in this paper, we present an illustrative application where the purpose is to compare distributions of self-assessed health in a group of ten European countries.

The paper lies in the intersection of three main areas of literature. Chiefly, the paper contributes to the literature on order relations defined on convex cones, starting with Marshall, Walkup and Wets (1967). This literature has further been specialized and extended by Magdalou (2021), who establishes the fundamental role the dual cone plays in deriving the underlying set of order-preserving functions. This is an important result, as it provides an analytical method of deriving a numerical representation of the relation, via a characterization of the extreme rays of the dual cone. The author however comments that this result (Proposition 9, Magdalou 2021) can be difficult to apply in practice, since "...In general, finding an analytical solution for the extreme points is far from trivial". In comparison, the criteria proposed in this paper are simple to implement, and draw on efficient computational methods from the linear programming literature. A further contribution, Abul Naga (2022), discusses instances where the numerical representation of the incomplete relation may be recovered from the Hilbert basis of the underlying cone. The present paper generalizes Abul Naga (2022) in that the same results may be obtained in a wider class than the so-called class of *maximal linearly independent Hilbert bases* considered by this author.

A second literature the paper builds on is the utility representation of incomplete preferences, starting with Aumann (1962), and Ok (2002), where our present focus is on obtaining new criteria for ordering distributions in the specific context of relations defined on finitely generated cones. A third literature the paper draws on is centred around Farkas lemma, and the characterization of *positive solutions* of systems of linear equations. There, we draw on results pertaining to generalized inverses of matrices (e.g. Ben-Israel and Greville, 2003 and Abadir and Magnus, 2005) in order to associate the comparability of a pair of distributions with the existence of positive solutions of a system of linear equations. Finally, from a purely practical angle, the paper draws on a literature related to computation with convex cones. Efficient algorithms for finding positive bases of cones are routinely required in the field of linear programming (Schrijver, 1986), and these algorithms also come handy for applying the criteria introduced in this paper for comparing distributions. To the best of our knowledge however, the use of generalized inverses and linear programming methods in order to provide criteria for ordering distributions, as done in this paper, is novel.

After reviewing key concepts and definitions in Section 2, the purpose of Section 3 of the paper is to characterize the finite set of simplicial relations such that one such relation in the set provides a criterion for investigating whether two distributions x and y are comparable. Section 4 contains an illustrative application of the methodology, while Section 5 concludes the paper. An appendix contains proofs of the main results of this paper.

2. Order relations and convex cones

We begin this section by defining order relations induced by convex cones. We next review some properties of finitely generated cones, that will be key to the approach we develop to investigate the comparability of pairs of distributions. The approach we will pursue in this section is to define a general relation $\succeq$ on a convex cone $\mathcal{C}$ in $\mathbb{R}^d$. Implicit in our discussion throughout the paper, is that each vector in $\mathcal{C}$ takes the form of a difference between two distributions pertaining to a variable defined on d ordered socioeconomic states. The discussion could equally be framed in terms of lotteries defined on a finite number of states of nature.

In what follows the sets $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively denote the integers, rationals and real numbers. We let $\mathbb{Z}_+ := \{0, 1, 2, ...\}$ denote the non-negative integers, and we likewise define the sets $\mathbb{Q}_+$ and $\mathbb{R}_+$. The notation a is used to denote a vector, A denotes a matrix, while $\mathcal{A}$ is taken to denote a set. Let u' denote the transpose of a vector $u \in \mathbb{R}^d$, and let $\mathbf{0}_d$ denote a vector of d zeroes. If $u = (u_1, \cdots, u_d)$ and $x = (x_1, \cdots, x_d)$ are two vectors in $\mathbb{R}^d$, we denote the inner product $u_1 x_1 + \cdots + u_d x_d$ by $u \cdot x$.

A relation $\succeq$ on $\mathbb{R}^d$ is called a *preorder* if it is transitive and reflexive, and a *partial ordering* if it is transitive, reflexive and antisymmetric [1]. A relation $\succeq$ is additive if for all $x, y, z \in \mathbb{R}^d$, $x \succeq y$ implies $x + z \succeq y + z$. Finally, the relation $\succeq$ is scale invariant if for all $x, y \in \mathbb{R}^d$, and for all $\lambda > 0$, there holds $x \succeq y$ implies $\lambda x \succeq \lambda y$. Following Marshall et al. (1967), an additive and scale invariant partial order relation $\succeq$ may be associated with a pointed convex cone [2] $\mathcal{C} \subseteq \mathbb{R}^d$, whereby $x \succeq y$ if and only if $x - y$ is a vector that belongs to the convex cone $\mathcal{C}$. Under such circumstances, we more simply refer to $\succeq$ as an order relation induced by a convex cone, or a cone ordering.

A cone $\mathcal{C}$ is said to be *finitely generated* if it arises as the *positive span* of a finite set of vectors. Particular types of finitely generated cones, known as simplicial cones, and the associated order relations, will play an important role in this paper. We define these concepts next.

Definition 1 *(i) Let $\mathcal{V} := \{v^1, \ldots, v^m\}$ denote a finite set of vectors in $\mathbb{R}^d$. The positive span of $\mathcal{V}$ is the set of all positive linear combinations of $v^1, \ldots, v^m$:*

$$\text{cone}(\mathcal{V}) := \left\{\lambda_1 v^1 + \cdots + \lambda_m v^m : \lambda_1, \ldots, \lambda_m \in \mathbb{R}_+\right\}, \tag{2.1}$$

and the set $\mathcal{V}$ is said to positively span a finitely generated cone $\mathcal{C}$ if $\text{cone}(\mathcal{V}) = \mathcal{C}$.

(ii) A cone $\mathcal{C} = \text{cone}(\{v^1, \ldots, v^m\}) \subseteq \mathbb{R}^d$ is said to be simplicial if $m = d$ and $v^1, \ldots, v^m$ are linearly independent vectors.

(iii) An order relation $\succeq$ on $\mathbb{R}^d$ is said to be simplicial if there is a simplicial cone $\mathcal{C} \subseteq \mathbb{R}^d$ such that $x \succeq y$ if and only if $x - y \in \mathcal{C}$ for all vectors x and y in $\mathbb{R}^d$.

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[1] A relation $\succeq$ on $\mathbb{R}^d$ is called *transitive* if $x \succeq y$ and $y \succeq z$ imply $x \succeq z$ for all $x, y, z \in \mathbb{R}^d$, *reflexive* if $x \succeq x$ for all $x \in \mathbb{R}^d$, and *antisymmetric* if $x \succeq y$ and $y \succeq x$ imply $x = y$ for all $x, y \in \mathbb{R}^d$.

[2] A cone $\mathcal{C}$ in $\mathbb{R}^d$ is said to be pointed if for all $x \in \mathcal{C}$ such that $x, -x \in \mathcal{C}$ there holds $x = \mathbf{0}_d$.

Let P denote a $d \times m$ real matrix and b a vector in $\mathbb{R}^d$. There are two important results related to cones that we shall make extensive use of, namely Farkas lemma and Carathéodory's theorem. First however, we recall some concepts and results related to the solvability of a system of linear equations of the form $Pv = b$ (where the unknown vector v need not be positive). We define the *rank* of P to be the number of linearly independent columns of P, and we write this number as $\text{rank}(P)$. The system $Pv = b$ is said to be *solvable* if there is a vector w in $\mathbb{R}^m$ such that $Pw = b$. Such a solution w exists if and only if P and the augmented matrix $\begin{pmatrix} P & b \end{pmatrix}$ have identical rank, equivalently if b is a linear combination of the columns of P. Under such conditions, it is possible to find an $m \times d$ matrix G such that $w = Gb$ is a solution of the system $Pv = b$. We call the matrix G a *generalized inverse* of P. We refer the reader to chapter 10 of Abadir and Magnus (2005) for further discussion on generalized inverses in relation to the solvability of systems of linear equations.

We next specialize the above discussion to investigating the existence of non-negative solutions of the system $Pv = b$. Geometrically, we are interested in conditions that establish that the vector b lies in the cone generated by the m columns of the matrix P. A first result is provided by Farkas lemma.

Lemma 1 (Farkas' lemma) *Let P denote a $d \times m$ real matrix and b a vector in $\mathbb{R}^d$. Then exactly one of the following alternatives hold*:

(1) *The system of linear equations $Pv = b$ has a solution $w \in \mathbb{R}^m_+$.*

(2) *There is a vector c in $\mathbb{R}^d$ such that $c'P \leq \mathbf{0}_m$ and $c \cdot b > 0$.*

Farkas lemma is useful to investigate the existence of *positive* solutions w in association with a system of linear equations $Pv = b$. The lemma states that exactly one of the two statements (1) and (2) must be true. If (1) holds, it is possible to express b as a positive combination of the columns from P (with weights given by the vector w). In the alternative (2), the system $Pv = b$ does not admit positive solutions, and there exists a vector c in d-dimensional space such that $c \cdot b > 0$ and $c \cdot p^i \leq 0$ for each of the m columns p^i of P. Algorithms are available for computing the vector c when the alternative (2) holds, and we shall discuss in Section 3 one such algorithm that appears in Schrijver (1986).

The second result we shall make extensive use of is known as Carathéodory's theorem.

Lemma 2 (Carathéodory's theorem) *Let $\mathcal{P} := \{p^1, \ldots, p^m\}$ denote a finite set of vectors in d-dimensional space $\mathbb{R}^d$, and define the cone $\mathcal{C} = \text{cone}(\mathcal{P})$. If b is* *a point in $\mathcal{C}$, then b belongs to a cone generated by a linearly independent subset of vectors from $\mathcal{P}$.*

In simple terms, Carathéodory's theorem says that if a vector b belongs to the cone $\mathcal{C}$, then b can be expressed as a linear combination of d linearly independent vectors from the set $\mathcal{P}$, with the property that the weights defining the linear combination are non-negative. This result, together with Farkas lemma, underlies the approach developed in this paper.

We now turn our attention to the comparison of certain types of vectors in $\mathbb{R}^k$, that we shall refer to as *distributions*. Let $\mathbb{D}_n^k := \{x \in \mathbb{Z}_+^k : x_1 + \cdots + x_k = n\}$, denote the set of distributions of a given sum total, defined on k ordered socioeconomic states, where $i = 1$ denotes the worst socioeconomic state, and $i = k$ indexes the highest state [3]. In the empirical application we shall consider for instance, the European statistical agency EUROSTAT collects data on self-assessed health, asking respondents in each participating country to choose one of five possible assessments: *very bad, bad, average, good, or very good.* The state $i = 1$ then corresponds to a very bad health, while $i = k$ pertains to a state of being in very good health.

When we consider a difference of vectors pertaining to a pair of distributions x and y in $\mathbb{D}_n^k$, the resulting vector $x - y$ belongs to the following subspace $\mathbb{S}^k$ of $\mathbb{R}^k$:

$$\mathbb{S}^k := \left\{ s \in \mathbb{R}^k : s_1 + s_2 + \cdots + s_k = 0 \right\}. \tag{2.2}$$

Because each vector in $\mathbb{S}^k$ sums to zero, it is important to observe that the maximum size of a linearly independent set in $\mathbb{S}^k$ is equal to $k-1$ (rather than k); that is, the subspace $\mathbb{S}^k$ has dimension equal to $k-1$.

The preferences of a social planner are assumed to be given by a relation $\succeq_T$, associated with a cone $\mathcal{C}_T \subseteq \mathbb{R}^k$. Following Magdalou (2021), the cone $\mathcal{C}_T$ is finitely generated by a set of vectors $\mathcal{T}$, known as the *set of transfers.* We can think of each vector in the set of transfers as providing a direction of increasing social preference. That is, a finite set of vectors

$$\mathcal{T} := \left\{ t^1, \ldots, t^m \right\} \tag{2.3}$$

is a set of transfers if for all $t \in \mathcal{T}$,

[T1] t can be written as the difference between two distributions in $\mathbb{D}_n^k$, and

[3] For instance, if $x^1 = (1\ 2\ 0\ 0\ 1)$, then x^1 is an element of $\mathbb{D}_4^5$, and we adopt the convention that the frequency distribution $x^2 = (1/4\ 2/4\ 0\ 0\ 1/4)$ is an element of $\mathbb{D}_1^5$.

[T2] $t \in \mathcal{T}$ implies $-t \notin \mathcal{T}$.

Observe from [T1] that each $t \in \mathcal{T}$ is a vector in $\mathbb{S}^k$, and from [T2] that the cone $\mathcal{C}_T := \text{cone}(\mathcal{T})$ is pointed. It follows therefore from [T1] and [T2] that the set of transfers $\mathcal{T}$ positively spans a pointed rational cone $\mathcal{C}_T := \text{cone}(\mathcal{T})$, associated with the relation $\succeq_T$. Furthermore, because the cone $\mathcal{C}_T$ is pointed, the matrix $T := \begin{pmatrix} t^1 & \cdots & t^m \end{pmatrix}$ whose columns are the m vectors t^i in the set of transfers, is a *maximum rank* matrix in $\mathbb{S}^k$. That is,

$$\text{rank}(T) = k - 1. \tag{2.4}$$

Finally, let x and y be two distributions in $\mathbb{D}_n^k$, such that for $\lambda_1, ..., \lambda_m \in \mathbb{R}_+$ and vectors $t^1, ..., t^m \in \mathcal{T}$, we can write $x - y = \sum_{s=1}^{m} \lambda_s t^s$. Then it is the case that the vector $x - y$ belongs to the cone $\mathcal{C}_T$, and that $x \succeq_T y$ [4].

Below, we provide an example of an order relation defined on a finitely generated cone, and the associated set of transfers. The relation was introduced by Gravel Magdalou and Moyes (2021), and was further discussed in Magdalou (2021).

Example 1 (The Hammond inequality order)

Let x, y be two distributions in $\mathbb{D}_n^k$. Following Gravel et al. (2021), we say that x is obtained from y via an *egalitarian Hammond transfer* if for indices $h < i \leq j < l$ in the index set $\{1, ..., k\}$ there holds $x_h = y_h - 1$, $x_i = y_i + 1$, $x_j = y_j + 1$, $x_l = y_l - 1$ and $x_m = y_m$ for all $m \neq h, i, j, l$. When $i = j$, this definition specializes a Hammond transfer to the form $x_h = y_h - 1$, $x_i = y_i + 2$, $x_l = y_l - 1$ and $x_m = y_m$ for all $m \neq h, i, l$.

When taking the simple case where there are $k = 4$ socioeconomic states, we may accordingly define the set of transfers associated with this inequality order as comprising the following five vectors:

$$\mathcal{T} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \\ 0 \end{pmatrix} \right\}. \tag{2.5}$$

[4] In the case where x and y are integral vectors in $\mathbb{D}_n^k$ and the set of transfers $\mathcal{T}$ contains a Hilbert basis of the cone $\mathcal{C}_T$, it is furthermore possible to obtain a representation $x - y = \sum_{s=1}^{m} \theta_s t^s$ with suitably chosen weights $\theta_1, ..., \theta_m \in \mathbb{Z}_+$. We refer the reader to Magdalou (2021), who defines such sets of transfers $\mathcal{T}$ as being *minimal*.

and we say that x is more egalitarian than y if $x - y$ is an element of the cone generated by the set of transfers, equivalently, if $x - y \in \text{cone}(\mathcal{T})$.

Now consider a hypothetical vector $z = (-3\ 2\ 5\ -4)'$. This vector is constructed as the sum $t^1 + t^2 + t^3 + t^4$, where t^i is the i-th vector in the set of transfers $\mathcal{T}$, and accordingly z is an element of the Hammond inequality order cone. $\Diamond$.

3. Comparing distributions

Consider a pair of distributions x and y in $\mathbb{D}_n^k$, and let $z = x - y$. Consider the linear system of equations

$$\begin{aligned} z &= \rho_1 t^1 + \cdots + \rho_m t^m \text{ with } t^1, \cdots, t^m \in \mathcal{T} \\ &= T\rho, \end{aligned} \tag{3.1}$$

where z is a k-dimensional vector given by the data, $T = (t^1, ..., t^m)$ is the matrix whose columns are the m elements of the set of transfers and $\rho = (\rho_1, \ldots, \rho_m)$ is a m-dimensional real vector of unknown coefficients. From (2.4), we have that T is a maximum rank matrix in the subspace $\mathbb{S}^k$. It follows therefore that for any $z \in \mathbb{S}^k$, there holds that $\text{rank}(T) = \text{rank}\begin{pmatrix} T & z \end{pmatrix} = k - 1$, so that the necessary and sufficient condition for the solvability of the linear system (3.1) is met for any such vector $z \in \mathbb{S}^k$. As such, there must exist a vector $\mu \in \mathbb{R}^m$ and a generalized inverse G of the matrix T such that $z = T\rho$ if and only if $Gz = \mu$. Our interest being in investigating whether $x \succeq_T y$, we seek however a *non-negative solution* $\mu \in \mathbb{R}_+^m$ of the linear system (3.1). Also, for the purpose of undertaking distributional comparisons in applied work, we shall require a form for the generalized inverse G.

Let $J := \{i_1, ..., i_{k-1}\}$ denote a subset of indices from the set $\{1, ..., m\}$, and define the following subset of the set of transfers:

$$\mathcal{T}_J := \left\{ t^{i_1}, \cdots, t^{i_{k-1}} : \ t^j \in \mathcal{T} \quad \text{for all } j \in J \right\}. \tag{3.2}$$

Construct the matrix $T_J := (t^{i_1} \ \cdots \ t^{i_{k-1}})$, whose $k - 1$ columns are the elements of the set $\mathcal{T}_J$. When the matrix T_J is of maximum rank in the subspace $\mathbb{S}^k$, equivalently when the vectors from the subset $\mathcal{T}_J$ are linearly independent, we say that the set of indices J is *simplicial*. We also define the cone

$$\mathcal{C}_{T_J} := \text{cone}\ (T_J), \tag{3.3}$$

and, following Definition 1, we say that $\mathcal{C}_{T_J}$ is a *simplicial* cone. We associate with $\mathcal{C}_{T_J}$ a relation $\succeq_{T_J}$ such that for any pair of distributions x and y in $\mathbb{D}_n^k$ there holds $x \succeq_{T_J} y$ if and only if $x - y$ is an element of the simplicial cone $\mathcal{C}_{T_J}$. Underlying the following result is Carathéodory's theorem.

Lemma 3 *Let x and y denote two distributions in $\mathbb{D}_n^k$. In relation to the statements below, the implications $(i) \Longrightarrow (ii) \Longrightarrow (iii)$ hold.*

(i) $x \succeq_T y$.

(ii) $x - y$ is a positive combination of a linearly independent subset of vectors from the set of transfers (2.3).

(iii) There is an order relation $\succeq_{T_J}$ on a simplicial cone $\mathcal{C}_{T_J} \subseteq \mathcal{C}_T$, such that $x \succeq_{T_J} y$.

Observe that because the cone $\mathcal{C}_{T_J}$ of Lemma 3 is simplicial, there results from Definition 1 (iii) that the relation $\succeq_{T_J}$ is a simplicial order relation. Observe also that because the cone $\mathcal{C}_{T_J}$ is constructed from a subset of $k - 1 \leq m$ vectors from the set of transfers, there results that the order relation $\succeq_{T_J}$ is less complete that $\succeq_T$. That is, there are instances where a given simplicial subset T_J does not positively span a vector z that belongs to the cone $\mathcal{C}_T$. We illustrate the above discussion with the following extension of Example 1.

Example 2 (The Hammond inequality order, continued)

Return to Example 1 that discusses the Hammond inequality order in the context of $k = 4$ socioeconomic states. We first construct the matrix T associated with the set of transfers (2.5) as follows

$$T = \begin{pmatrix} -1 & 0 & -1 & -1 & -1 \\ 1 & -1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix} \tag{3.4}$$

There are $\dfrac{5!}{3!\ 2!} = 10$ sets J comprising $k - 1 = 3$ indices from the index set $\{1, 2, 3, 4, 5\}$, pertaining to the columns of the matrix T. These are $J_1 = \{1, 2, 3\}$, $J_2 = \{1, 2, 4\}$, $J_3 = \{1, 2, 5\}$, $J_4 = \{1, 3, 4\}$, $J_5 = \{1, 3, 5\}$, $J_6 = \{1, 4, 5\}$, $J_7 = \{2, 3, 4\}$, $J_8 = \{2, 3, 5\}$, $J_9 = \{2, 4, 5\}$ and $J_{10} = \{3, 4, 5\}$. Of these, two sets of indices, J_3 and J_4, define matrices

$$T_{J_3} = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad T_{J_4} = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \\ -1 & -1 & -1 \end{pmatrix} \tag{3.5}$$

that have rank 2. The remaining sets of indices are *simplicial*, in that they enable us to construct matrices of rank 3.

Now return to the vector $z = (-3\ 2\ 5\ -4)'$, constructed as the sum $t^1 + t^2 + t^3 + t^4$, where t^i is the i-th column of the matrix T. By running through the eight simplicial sets, we find that z is positively spanned by sets of the form $\{t^j : j \in J\}$ for each of the simplicial sets J_1, J_2, J_7, J_8, J_9. For instance, using J_1, we find that z belongs to the simplicial cone spanned by the first three vectors from the set of transfers (2.5), and we can express z in the form $z = 3t^1 + t^2 + 0t^3$. However, because $z = 4t^1 + 0t^3 - t^5$, we cannot express z as a positive combination of the vectors t^1, t^3, and t^5 that span the cone associated with the simplicial set J_5. It is in this sense that we mean in Lemma 3 that a given relation $\succeq_{T_J}$, defined on a simplicial cone $\mathcal{C}_{T_J}$, is less complete that $\succeq_T$. $\Diamond$

Let $J = \{i_1, ..., i_{k-1}\}$ denote a subset of indices from the set $\{1, ..., m\}$, and define the subset $\mathcal{T}_J := \{t^{i_1}, ..., t^{i_{k-1}}\} \subseteq \mathcal{T}$. We associate with the set of transfers $\mathcal{T}$ a finite set of cones $\mathcal{K}_T$ in $\mathbb{R}^k$, such that any cone $\mathcal{C} \in \mathcal{K}_T$ arises as the positive span of a subset of $k-1$ linearly independent vectors taken from the set of transfers $\mathcal{T}$:

$$\mathcal{K}_T := \left\{\mathcal{C}_{T_J} = \text{cone } (\mathcal{T}_J) : t^{i_1}, ..., t^{i_{k-1}} \in \mathcal{T} \text{ are linearly independent}\right\} \tag{3.6}$$

Observe from Definition 1 (ii) that $\mathcal{K}_T$ is the set of simplicial cones associated with the set of transfers. Because the cone $\mathcal{C}_T$ is pointed, there is at least one element in the set $\mathcal{K}_T$. Furthermore, there are at most $\dfrac{m!}{(m-k+1)!(k-1)!}$ simplicial cones in $\mathcal{K}_T$, corresponding to the number of ways of choosing $k-1$ vectors from the set of transfers $\mathcal{T}$. As such, the set of simplicial cones $\mathcal{K}_T$ is non-empty and finite. As we shall see in Corollary 1 below, the simplicial cones in $\mathcal{K}_T$ will provide a numerical criterion for investing whether two distributions x and y are comparable.

Let $\widehat{T}$ denote a $(k-1) \times m$ matrix constructed from the vectors of the set of transfers, and let $\widehat{z}$ denote a $(k-1)$-vector obtained from the vector $x - y$ [5]. The

[5] Exact expressions of the matrix $\widehat{T}$ and vector $\widehat{z}$ will be provided below.

next result provides an equivalence between four statements: (i) the fact that two distributions x and y are ordered by the relation $\succeq_T$ defined on a finitely generated cone $\mathcal{C}_T$, (ii) the possibility of expressing $x-y$ as a positive combination of a subset of linearly independent vectors from the set of transfers, (iii) the existence of a simplicial relation $\succeq_{T_J}$ such that x and y are ordered by this latter relation, and (iv) the existence of a generalized inverse G of the matrix $\widehat{T}$, such that the product of G and $\widehat{z}$ is greater or equal to $\mathbf{0}_{k-1}$. As a byproduct of Theorem 1 below, a form for the generalized inverse G to be used in applied work will be provided subsequently [6].

To state the result compactly, we make use of the following notation: for any vector $a = (a_1, ..., a_k)'$, the vector $\widehat{a} = (a_1, ..., a_{k-1})'$ will denote the projection of a on its $k-1$ first coordinates. That is, we shall let $\widehat{x}$ denote the vector $(x_1, ..., x_{k-1})$, $\widehat{y} = (y_1, ..., y_{k-1})$ and, in relation to the set of transfers $\mathcal{T} = \{t^1, ..., t^m\}$, $\widehat{T}$ will denote the matrix whose columns are given by the vectors $\widehat{t^1}, \dots, \widehat{t^m}$.

Theorem 1 *Let $\mathcal{T} = \{t^1, ..., t^m\} \subseteq \mathbb{Z}^k$ denote a set of transfers that contains a Hilbert basis. Then, the following statements are equivalent:*

(i) $x \succeq_T y$.

(ii) *$x - y$ is a positive combination of a linearly independent subset of vectors from the set of transfers.*

(iii) *There is a simplicial cone $\mathcal{C}_{T_J} \subseteq \mathcal{C}_T$ and a simplicial relation $\succeq_{T_J}$ defined on $\mathcal{C}_{T_J}$, such that $z \in \mathcal{C}_{T_J}$ and $x \succeq_{T_J} y$.*

(iv) *There is a generalized inverse G of the matrix $\widehat{T}$ and a vector $\theta \in \mathbb{R}^m_+$ such that $G(\widehat{x} - \widehat{y}) = \theta$.*

The significance of the result is as follows. As illustrated in Example 2, when $x \succeq_T y$, from statement (ii) of the theorem it is always possible to find a positive basis, that is a linearly independent subset from the set of transfers, such that this subset positively spans the vector $x - y$. In turn in (iii), the simplicial relation $\succeq_{T_J}$ associated with the resulting positive basis is coarser than the order relation $\succeq_T$, in the sense that the former only enables the researcher to order a subset of the pairs of distributions that are comparable in the latter relation. Furthermore, as we shall observe in Section 4.2, the same positive basis provides all the information needed in order to construct the generalized inverse G of the matrix T whose columns are the elements of the set of transfers. The generalized

[6] See Section 4.2, for an illustrative application, as well as the Appendix for general form of the generalized inverse constructed in this paper and its properties.

inverse in statement (iv) in turn provides a new empirical criterion, simple to implement, in order to investigate the comparability of a pair of distributions x and y.

As a first corollary to the theorem, we next turn to the numerical representation of the simplicial order relation $\succeq_{T_J}$. This utility representation is readily derived by inverting a matrix whose columns are constructed from the generators of the simplicial cone $\mathcal{C}_{T_J}$.

Corollary 1 *Let x and y denote two distributions in $\mathbb{D}_n^k$ such that $x \succeq_T y$, and take a simplicial cone $\mathcal{C}_{T_J} = \left\{T_J\alpha :\ \alpha \in \mathbb{R}_+^{k-1}\right\} \subseteq \mathcal{C}_T$ such that $(x-y) \in \mathcal{C}_{T_J}$. Then, the matrix $A =: -(\widehat{T}_J)^{-1}$ provides a numerical representation of the relation $\succeq_{T_J}$, in the sense that $A(\widehat{x} - \widehat{y}) \leq \mathbf{0}_{k-1}$ if and only if $(x - y) \in \mathcal{C}_{T_J}$.*

The above result is of practical significance in applied work. One way to construct the simplicial relation $\succeq_{T_J}$ that enables the researcher to order x and y, is to repeat the steps detailed in Example 2. Namely, one runs through the simplicial sets $J_1, ..., J_l$ until a simplicial set J is found such that $(x-y)$ is positive combination of the columns of $\widehat{T}_J$. Then, from Theorem 1, we have that $x \succeq_{T_J} y$. Because the columns of $\widehat{T}_J$ are linearly independent, $\widehat{T}_J$ is the required positive basis. In turn the inverse matrix $A =: -(\widehat{T}_J)^{-1}$ exists, and $A(\widehat{x} - \widehat{y}) \leq \mathbf{0}_{k-1}$. It is in this sense that the matrix A provides a multi-utility representation of the simplicial relation $\succeq_{T_J}$.

Within this framework, we next obtain a criterion that is equivalent to asserting that two distributions x and y are *in*comparable by the relation $\succeq_T$. As this condition will prove useful in the context of the illustrative application below, and more generally in applied work, this is stated below as a second corollary of Theorem 1.

Corollary 2 *Using the notation of Theorem* 1*, the following statements are equivalent*:

(i) *The distributions x and y are not comparable according to the relation $\succeq_T$.*

(ii) *There is a vector $c \in \mathbb{R}^{k-1}$ such that $c \cdot \widehat{z} > 0$ and $c \cdot \widehat{t^i} \leq 0$ for every vector t^i in the set of transfers $\mathcal{T}$.*

The corollary provides a new criterion for investing the *incomparability* of a pair of distributions. The result arises as an instance of Farkas lemma's second alternative (Lemma 1), in the context of a linear system of inequalities. That is, when the there exists a vector c in (ii) that separates the point $\widehat{z}$ and the cone spanned by the columns of the matrix $\widehat{T}$, Farkas' lemma informs us that the system of equations $\widehat{T}\mu = \widehat{z}$ does *not* possess non-negative solutions. In turn, x and y cannot be ordered by the relation $\succeq_T$.

In the case where the number of vectors in the set of transfers is large in comparison to the number of socioeconomic states the distributions x and y are constructed from, an important algorithmic question arises as how to compute the positive basis, that is the $k-1$ vectors in the set of transfers that positively span the vector $x-y$. This information is needed to construct the simplicial cone $\mathcal{C}_{T_J}$ in statement (*iii*) of Theorem 1, as well as the generalized inverse G in statement (*iv*). Likewise, the construction of the simplicial cone $\mathcal{C}_{T_J}$ is required to obtain the numerical representation of the simplicial relation $\succeq_{T_J}$ in Corollary 1. In the alternative, when x and y are not comparable, one may want to compute the vector c in Corollary 2, that yields the inequalities $c \cdot \widehat{z} > 0$ and $c \cdot \widehat{t^i} \leq 0$ for every $t^i \in \mathcal{T}$. Both of the above types of computations are readily implementable from an algorithm proposed in Schrijver (1986) in the context of linear programming. The algorithm is adapted in the appendix for the purpose of comparing distributions defined on a finitely generated cone.

4. An illustrative application

To illustrate the approach developed in this paper, we shall consider an empirical application pertaining to distributions of self-assessed health in a group of ten European countries from the 2017 wave of the EUROSTAT SILC database: Greece (GR), Spain (SP), France (FR), Italy (IT), Malta (ML), Belgium (BE), Germany (GER), Netherlands (NL), United Kingdom (UK) and Denmark (DK). EUROSTAT asks respondents in each participating country to rate their health according to a scale consisting of five ordered states. That is, $k = 5$ and responds choose one of five possible assessments: *very bad, bad, average, good, or very good.* The frequency distributions pertaining to the set of countries are reported in Table 1.

Within the context of the Hammond inequality order (Gravel et al. 2021; see also Examples 1 and 2 above), the purpose of the empirical illustration is to compare distributions using three different methods: (1) the numerical implementation criterion detailed in Theorem 5 of Gravel et al. (2021), (2) the empirical criteria of Theorem 1 of this paper, and (3) the use of Schrijver's linear programming algorithm to produce the vector c of Corollary 2 that separates the Hammond inequality order cone and the vector $\widehat{z} = \widehat{x} - \widehat{y}$ when x and y are not comparable.

In the context of this illustrative application where $k = 5$, it is useful to first define the vectors $\widehat{z} = (x_1 - y_1, ..., x_4 - y_4)$ and $\widetilde{z} = (x_2 - y_2, ..., x_5 - y_5)$. Theorem 5 of Gravel et al. (2021) enables us to conclude in the context of this application that x is more egalitarian than y if and only if the following two sets of partial sums inequalities are satisfied:

$$B_1\hat{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_3 - y_3 \\ x_4 - y_4 \end{pmatrix} \leq \mathbf{0}_4 \tag{4.1}$$

$$B_2\tilde{z} = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 - y_2 \\ x_3 - y_3 \\ x_4 - y_4 \\ x_5 - y_5 \end{pmatrix} \leq \mathbf{0}_4 \tag{4.2}$$

Together, these eight inequalities define a numerical implementation criterion for the Hammond inequality order.

In Table 2, we use the partial sums defined by the matrices B_1 and B_2 to report the outcome of comparing the 45 pairs of distributions using the Hammond inequality order. A 0 in row i and column j indicates that distributions pertaining to countries i and j are not comparable. A 2 entry in the (i, j) cell of Table 2 indicates that the distribution of row i is more egalitarian, while a -2 indicates that it is less egalitarian than the distribution pertaining to row j. Thus, the UK distribution is more egalitarian that of Greece (GR), not comparable to that of France (FR), and less egalitarian than that of Malta (ML).

4.1. The simplicial cones of the Hammond inequality order

Next consider comparing the same group of countries using relations defined on the simplicial cones of the Hammond inequality order. A particular type of set of transfers for the cone associated with the Hammond inequality order, known as a *minimal Hilbert basis*, is given by the columns of the following matrix (Abul Naga, 2022; Proposition 6)

$$T = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 2 & 2 & 0 & 2 & 2 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 2 & -1 & 0 & 2 & 2 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 \end{pmatrix} \tag{4.3}$$

When running through all $\frac{10!}{6!4!} = 210$ subsets of four vectors from the columns of T, we are able to construct 191 sets of linearly independent vectors. That is, when there are $k = 5$ health categories and there are $m = 10$ generators of the Hammond inequality order cone, the set $\mathcal{K}_T$ of (3.6) contains 191 simplicial cones.

When country x is more egalitarian than country y according to the Hammond inequality order, Theorem 1 states that (1) there is a simplicial cone $\mathcal{C}_{T_J} = \text{cone}(T_J)$ such $z = (x - y) \in \mathcal{C}_{T_J}$ and, (2) there is a generalized inverse G of $\widehat{T}$ and vectors $\mu, \theta \in \mathbb{R}^m_+$ such that $z = T\mu$ if and only if $G\widehat{z} = \theta$. For the comparison between the UK and Greece, there are seven such simplicial cones that positively span the vector z, and twenty for the comparison between the UK and Malta. For the remaining comparable pairs of distributions, the number of simplicial cones that positively span the vector z ranges between this lower bound of seven and the upper bound of twenty. For instance, for the comparison between Malta and Greece, we identify fifteen simplicial cones that generate the underlying vector z.

4.2. The generalized inverse of the matrix $\widehat{T}$

To construct a generalized inverse in relation to statement (*iv*) of Theorem 1, consider a positive basis for $x - y$, i.e. a set of four linearly independent vectors $\{t^i, t^j, t^k, t^l\}$ that positively span this vector. Let J denote the set of indices $\{i, j, k, l\} \subset \{1, ..., 10\}$. Next construct the matrix M by inverting the matrix $\widehat{T}_J = (\widehat{t}^i \ \widehat{t}^j \ \widehat{t}^k \ \widehat{t}^l)$, and let $m_1, ..., m_4$ denote the four rows of M. Note that in the context of (4.3), $\widehat{T}$ has four rows and ten columns. Accordingly, a generalized inverse G of $\widehat{T}$ has ten rows and four columns. For the purpose of undertaking distributional comparisons, a convenient choice of generalized inverse G of the matrix $\widehat{T}$ may be constructed as follows. For row h of G, set $g_h = (0, 0, 0, 0)$ if $h \in \{1, ..., 10\} \backslash J$ and set $g_h = m_h$ when $h \in J$.

For instance, in relation to (4.3), let the vector z be positively spanned by the columns of the matrix T_J where $J = \{3, 8, 9, 10\}$. That is, the third, eighth, nineth and tenth column of the matrix (4.3) provide a positive basis for z. To construct the required generalized inverse G, we first construct the matrix M as follows:

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1/2 \\ -4 & -2 & -1 & -1/2 \\ 4 & 2 & 1 & -1 \end{pmatrix} \tag{4.4}$$

Next, we may construct the generalized inverse G of $\widehat{T}$ as follows [7]:

$$G = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 & -4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & -1/2 & -1 \end{pmatrix}' \tag{4.5}$$

One point worth emphasizing, from the perspective of this illustrative application, is that all eight vectors $z = x - y$ of Table 2 that belong to the Hammond inequality order cone, jointly belong to a common simplicial cone $\mathcal{C}_{T_J}$. That is, we have found that, for all comparable pairs of distributions, the vector $z = x - y$ is spanned by a common simplicial cone defined by the set of vectors $\left\{\hat{t}^j : j \in \{3, 8, 9, 10\}\right\}$. In turn, the relation $\succeq_{T_J}$ defined on the simplicial cone $\mathcal{C}_{T_J}$ where $J = \{3, 8, 9, 10\}$, may uniquely be used to order the eight comparable pairs of distributions of Table 2. Furthermore, the generalized inverse (4.5) may uniquely be used to solve for a vector $\theta \in \mathbb{R}^{10}_{+}$ such that $G(\widehat{x} - \widehat{y}) = \theta$, in the context of all pairs of distributions that are comparable. Finally, (minus one times) the matrix M of (4.4) provides a numerical representation of the simplicial relation defined on cone$\{t^3,\, t^8,\, t^9,\, t^{10}\}$, in the sense of Corollary 1.

4.3. Results from Schrijver's algorithm

From comparable pairs of distributions, we turn our attention next to incomparable pairs. As discussed above, the distributions of self-assessed health pertaining to France and the UK are not comparable. From the above discussion, we may consider three distinct ways to reach this conclusion: (1) the occurrence a violation of one or more inequalities (4.1–4.2) from the criterion of Theorem 5 of Gravel et al. (2021), (2) the impossibility of finding a positive basis for the vector z and, (3) the existence of a vector c in Corollary 2 that separates the Hammond order cone from the vector z.

When comparing the France and UK distribution using the first of these approaches, we find that the first of the four inequalities in (4.1) is violated. Using the second approach, it is the case that we cannot find a simplicial cone in the set $\mathcal{K}_T$ of (3.6) that contains the vector z. Finally, in relation to (3), we appeal to Schrijver's algorithm to compute the required vector c. This produces the vector $c = (1\ 0\ \ 0\ \ 0)$, with the result that $c \cdot \widehat{z} = 0.01$, and $c \cdot \widehat{t^{\imath}} \in \{-1\ ,\ 0\}$ for every

[7] Observe in (4.5) below that as a result of transposition, the hth row of G is the hth column of the matrix that appears on the right-hand side of the equality sign.

column vector t^i of the Hilbert basis matrix (4.3) [8]. That is the vector c yields the inequalities $c \cdot \widehat{z} > 0$ and $c \cdot \widehat{t^i} \leq 0$ for every $t^i \in \mathcal{T}$ as required in Corollary 2. As such, either of the above three criteria enables us to conclude that the UK and France distribution are not comparable by the Hammond order relation.

5. Conclusions

In empirical investigations, the adoption of incomplete preference relations is often complicated by the fact that they require a derivation of a numerical representation, in order to enable the researcher to compare distributions, lotteries, or more generally vectors in the underlying choice set. The purpose of this paper was to exploit the geometry of order relations defined on finitely generated cones in order to obtain equivalent criteria for ordering distributions – criteria that do not require the prior derivation of a numerical representation of the order relation. We conclude by mentioning some limitations of the approach developed in this paper.

In relation to statement (*iii*) of Theorem 1, it would have been preferable to partition the cone $\mathcal{C}_T$ into simplicial subsets, such that each vector z belongs to a *unique* simplicial cone. This is however an active research area of computational geometry and the theory of triangulations, where the development of efficient algorithms for this purpose is still in its infancy (see for instance De Loéra, Rambau and Santos, 2010).

We also note that the number of simplicial cones that may be constructed from the set of transfers grows exponentially when the number of vectors in this set is much larger than the number of states the distribution is defined on. In such cases however, it may be possible to appeal to linear programming methods, such as Schrijver's algorithm, in order to investigate whether a pair of distributions are ordered.

6. Appendix

This appendix contains proofs of Lemma 3, Theorem 1, and presents an adaptation of Schrijver's linear programming algorithm for the purpose of computing the vectors from the set of transfers that positively span a vector $z = x - y$.

[8]Observe that the vector c that separates $\widehat{z}$ from cone$(\widehat{T})$, in this example, is equal to the first row of the matrix B_1 of (4.1).

Proof of Lemma 3 Consider first the implication $(i) \Longrightarrow (ii)$. Accordingly, assume that $x \succeq_T y$. From Carathéodory's theorem, there is a simplicial set of indices $J = \{i_1, ..., i_{k-1}\}$ and a matrix $T_J := (t^{i_1} \ \cdots \ t^{i_{k-1}})$ of full rank $k-1$, such that $x - y = T_J\lambda$, with $\lambda \in \mathbb{R}_+^{k-1}$. Therefore $x - y$ is a positive combination of $t^{i_1}, \ \cdots \ , \ t^{i_{k-1}}$, a linearly independent subset of vectors from the set of transfers. This establishes that $(i) \Longrightarrow (ii)$.

To prove $(ii) \Longrightarrow (iii)$, we construct the cone $\mathcal{C}_{T_J} := \left\{T_J\alpha : \ \alpha \in \mathbb{R}_+^{k-1}\right\}$, and observe from the proof of the first implication above, that the vector $x - y$ is an element of the cone $\mathcal{C}_{T_J}$. Because the $k-1$ columns of T_J are linearly independent, it follows that is $\mathcal{C}_{T_J}$ is a simplicial cone. Letting $\succeq_{T_J}$ denote the order relation defined on the cone $\mathcal{C}_{T_J}$ it follows that $x \succeq_{T_J} y$. Finally, because the vectors $t^{i_1}, \ \cdots \ , \ t^{i_{k-1}}$ are elements of the set of transfers $\mathcal{T}$, it follows that $\mathcal{C}_{T_J} \subseteq \mathcal{C}_T$. □

Proof of Theorem 1 In order to prove the equivalence between the three statements, we proceed to show that $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv) \Longrightarrow (i)$.

$(i) \Longrightarrow (ii) \Longrightarrow (iii)$ These implications were established in Lemma 3.

$(iii) \Longrightarrow (iv)$ Assume that there is a simplicial cone $\mathcal{C}_{T_J} \subseteq \mathcal{C}_T$ such that $z \in \mathcal{C}_{T_J}$. Without loss of generality, partition the columns of $\widehat{T}$ as $\widehat{T} = (V \ W)$, where V is the set of columns that define the simplicial cone $\mathcal{C}_{T_J}$ that spans the vector $\widehat{z}$. That is, $\mathcal{C}_{T_J} = \text{cone}\{V\}$ and

$$\widehat{z} = V\lambda + W\mathbf{0}_{m+1-k} \tag{6.1}$$

where $\lambda \in \mathbb{R}_+^{k-1}$ and V is a square matrix of maximum rank, equal to $k-1$. Let $\mathbf{0}^1$ and $\mathbf{0}^2$ denote respectively $(m+1-k) \times (k-1)$ and $(m+1-k) \times (m+1-k)$ matrices of zeroes. We may construct one such generalized inverse of $\widehat{T}$ follows:

$$G = \begin{pmatrix} V^{-1} \\ \mathbf{0}^1 \end{pmatrix},$$

where we note that G has m rows and $(k-1)$ columns. Then,

$$\begin{aligned} & \widehat{z} = V\lambda + W\mathbf{0}_{m+1-k} \\ \Longleftrightarrow \quad & G\widehat{z} = \left(\begin{pmatrix} V^{-1} \\ \mathbf{0}^1 \end{pmatrix} (V \ \ W)\right) \begin{pmatrix} \lambda \\ \mathbf{0}_{m+1-k} \end{pmatrix} \\ \Longleftrightarrow \quad & G\widehat{z} = \begin{pmatrix} I_{k-1} & V^{-1}W \\ \mathbf{0}^1 & \mathbf{0}^2 \end{pmatrix} \begin{pmatrix} \lambda \\ \mathbf{0}_{m+1-k} \end{pmatrix} \\ = \quad & \begin{pmatrix} \lambda + \mathbf{0}_{k-1} \\ \mathbf{0}_{m+1-k} \end{pmatrix}, \text{ a vector } \theta \in \mathbb{R}_+^m. \end{aligned}$$

To complete the proof, we explain in what way G is a generalized inverse of $\widehat{T}$. It is readily verified that the following identities hold:

$$\begin{aligned}\widehat{T}G\widehat{T} &= \widehat{T}\\ G\widehat{T}G &= G\\ \widehat{T}G &= I_{k-1}, \quad \text{a symmetric matrix.}\end{aligned}$$

As such, G is a type $\{1,2,3\}$ inverse of $\widehat{T}$, in the sense of Ben-Israel and Greville (2013). We note however that G is not a Moore-Penrose inverse, as the latter would further require that $G\widehat{T}$ be a symmetric matrix.

$(iv) \Longrightarrow (i)$ Let G denote a generalized inverse of T and $\theta \in \mathbb{R}^m_+$, such that $Gz = \theta$. We wish to show that z is an element of the discrete cone generated by the set of transfers $\mathcal{T}$, or equivalently that $x \succeq_T y$. To this effect, note that because G is a generalized inverse of T and $Gz = \theta$, θ is a positive solution of the linear system $T\mu = z$. Therefore, it follows that $T\theta = z$; that is, $z = \theta_1 t^1 + \cdots + \theta_m t^m$, where $\theta_1, ..., \theta_m \geq 0$. In turn, we may conclude that z is an element of the rational cone $\mathcal{C}_T = \text{cone}\{\mathcal{T}\}$. Because z is an integral vector of the cone $\mathcal{C}_T$, and the set of transfers contains a Hilbert basis, it follows that there are integers $\gamma_1, ..., \gamma_m \in \mathbb{Z}_+$ such that $z = \gamma_1 t^1 + \cdots + \gamma_m t^m$, equivalently, that $x \succeq_T y$. □

6.1. Schrijver's algorithm

We next present the sets of inputs, outputs and the computations involved in the adaptation of Schrijver's algorithm in the context of this paper.

Input $\widehat{\mathcal{T}} = \{\widehat{t^1}, \cdots, \widehat{t^m}\} \subseteq \mathbb{R}^{k-1}$, $\widehat{z} \in \mathbb{R}^{k-1}$, $\mathbb{N}_m = \{1, ..., m\}$ and $\mathcal{J}^o = \{i_1, ..., i_{k-1}\} \subseteq \mathbb{N}_m$, a set of indices pertaining to a maximal set of linearly independent vectors from $\widehat{\mathcal{T}}$, and the matrix $\widehat{T} = (V\ W)$ where $V = \left(\widehat{t^{i_1}}, \cdots, \widehat{t^{i_{k-1}}}\right)$.

Output A vector $\mu \subseteq \mathbb{R}^m_+$ such that $\hat{z} = \widehat{T}\mu$ or a vector $c = (c_1, ..., c_{k-1})$ such that $c\hat{z} > 0$ and $cv^j \leq 0$ for all $j \in \mathbb{N}_m$.

Step 1 Compute $\lambda_{i_1}, .., \lambda_{i_{k-1}}$ such that $\widehat{z} = \lambda_{i_1}\widehat{t^{i_1}} + ... + \lambda_{i_{k-1}}\widehat{t^{i_{k-1}}}$. If $\lambda_{i_1}, .., \lambda_{i_{k-1}} \geq 0$, set $\mu_i = 0$ for all $i \notin \mathcal{J}^o$ and $\mu_i = \lambda_i$ for all $i \in \mathcal{J}^o$. Then $\mu \in \mathbb{R}^m_+$ and $\widehat{z} \in \text{cone}(\widehat{\mathcal{T}})$ and proceed to terminate at Step 6. If $\lambda \notin \mathbb{R}^{k-1}_+$, proceed to Step 2.

Step 2 Pick the smallest index $h \in \mathcal{J}^o$ with $\lambda_h < 0$. Construct the matrix M with columns v^j, with $j \in \mathcal{J}^o$, and the row vector $f = (f_1, ..., f_{k-1})$ given by the h-th row of the matrix M^{-1}. Then $fv^h = 1$, $fv^j = 0$ for all $j \in \mathcal{J}^o \backslash \{h\}$, and $f\widehat{z} < 0$.

Step 3 Compute $f\widehat{t^j}$ for all $j \in \mathbb{N}_m$. If $f\widehat{t^j} \geq 0$ for all $j \in \mathbb{N}_m$, and given that from Step 2 $f\widehat{z} < 0$ we conclude from Farkas lemma that $\widehat{z} \notin \text{cone}(\widehat{\mathcal{T}})$, and we proceed to terminate in Step 6. Otherwise we proceed to Step 4.

Step 4 Pick the smallest index $l \in \mathbb{N}_m$ such that $f\widehat{t^l} < 0$. Remove the index h from $\mathcal{J}^o$, replacing it by l, and define the new set of indices $\mathcal{J}^1 = (\mathcal{J}^o \backslash \{h\}) \cup \{l\}$.

Step 5 Set $\mathcal{J}^o := \mathcal{J}^1$ and return to Step 1.

Step 6 Terminate. Output the vector $\mu \subseteq \mathbb{R}^m_+$ if $\widehat{z} \in \text{cone}(\widehat{\mathcal{T}})$ and the vector $f = (f_1, ..., f_{k-1})$ otherwise.

By setting $c = -f$, we therefore obtain the required vector such that $c \cdot \widehat{z} > 0$ and $c\widehat{t^j} \leq 0$ for all $j \in \mathbb{N}_m$, in the case where $\widehat{z} \notin \text{cone}(\widehat{\mathcal{T}})$.

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Table 1: The distributions of self-assessed health in ten European countries

Country	Very bad	bad	average	good	Very good
GR	0.0230	0.0809	0.1548	0.2897	0.4515
SP	0.0140	0.0511	0.1922	0.5516	0.1912
FR	0.0100	0.0730	0.2430	0.4300	0.2440
IT	0.0080	0.0500	0.1720	0.6340	0.1360
ML	0.0050	0.0360	0.2050	0.4670	0.2870
BE	0.0150	0.0710	0.1690	0.4370	0.3080
GER	0.0170	0.0679	0.2607	0.4685	0.1858
NL	0.0080	0.0380	0.1930	0.5370	0.2240
UK	0.0160	0.0549	0.1808	0.3946	0.3536
DEN	0.0180	0.0601	0.2092	0.4565	0.2563

Table 2: Inequality comparison of countries by the Hammond inequality order

	GR	SP	FR	IT	ML	BE	GER	NL	UK	DEN
GR	=									
SP	0	=								
FR	0	0	=							
IT	0	2	2	=						
ML	2	0	0	0	=					
BE	0	0	0	0	0	=				
GER	0	0	0	0	0	0	=			
NL	0	0	0	0	0	2	0	=		
UK	2	0	0	0	-2	0	0	0	=	
DEN	0	-2	0	-2	0	0	0	0	0	=