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Which way up? Consistency, anti-consistency and inconsistency of social welfare and inequality partial orderings for ordinal data

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Abstract

In the context of distributional comparisons, we introduce consistency and anti-consistency as two properties for partial orderings defined on ordinal variables. A consistent social welfare or inequality partial ordering regards distribution $\mathbf{p}$ more desirable than $\mathbf{q}$ if and only if it ranks the corresponding reverse-ordered distribution $\mathbf{Rp}$ more desirable than $\mathbf{Rq}$. An anti-consistent partial ordering regards $\mathbf{p}$ more desirable than $\mathbf{q}$ if and only if the reverse-ordered distribution $\mathbf{Rp}$ is less desirable than $\mathbf{Rq}$. First, for a broad class of social welfare and inequality partial orders, which we call linear, we characterise those relations which are robust to any given type of permutation or reversal of the categories. Deploying these results as a specific consistency test for some prominent examples in the literature, we demonstrate the consistency of the Hammond inequality order and establish the anti-consistency of first-order dominance, and the inconsistency of two forms of the Hammond welfare partial ordering. Then deploying consistency tests based on dominance implementation criteria, we show that, among relations not falling in the linear class, the median-preserving spreads and the bipolarisation partial orderings are both consistent, whereas the status Lorenz ordering is inconsistent.

JEL codes: D63, I14, I24, I31 I32..
Keywords: Ordinal variables, social welfare, inequality, partial orderings, consistency.

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1. Introduction

Generally, the concern for consistency in distributional comparisons arises when there is more than one admissible way to present the data. For example, continuous bounded variables admit two alternative representations, namely attainments or shortfalls. Though seemingly trivial, the choice between these two defensible alternative representations bears serious normative and empirical implications, which have been thoroughly studied.[1] The normative question, then, is whether the consistency of distributional comparisons, i.e., rankings which are robust to these alternative representations, is desirable itself. Recently, Yalonetzky (2022) argued that this same challenge is present with ordinal variables since, without any loss of relevant information, their categories can be sorted in either ascending or descending order. Think about self-reported health, life satisfaction, education levels, sanitation ladders, political preferences or religious observance. For example, the Pew Research Center in the US has measured political ideology using six ordered categories (Center, 2014b,a). Hence, if we wanted to study trends in ideological polarisation in the US, should the assessment be sensitive to the choice between the order 'consistently liberal', 'mostly liberal', 'mixed', 'mostly conservative', 'consistently conservative', and the reverse ordering of these categories?

Consider, for instance, two comparable distributions $\mathbf{p} = (0.1, 0.3, 0.2, 0.1, 0.3)$ and $\mathbf{q} = (0.05, 0.4, 0.2, 0, 0.35)$, and let $I(\mathbf{p}) > I(\mathbf{q})$ for some inequality index I. Now consider their respective reverse-ordered distributions: $\mathbf{Rp} = (0.3, 0.1, 0.2, 0.3, 0.1)$ and $\mathbf{Rq} = (0.35, 0, 0.2, 0.4, 0.05)$. Should we demand $I(\mathbf{Rp}) > I(\mathbf{Rq})$? Yalonetzky (2022) introduced the consistency property for inequality indices applied to ordinal data, whereby for a given index I and every pair of comparable distributions $\mathbf{p}$ and $\mathbf{q}$, $I(\mathbf{p}) \geq I(\mathbf{q}) \leftrightarrow I(\mathbf{Rp}) \geq I(\mathbf{Rq})$. But Yalonetzky (2022) did not explore the challenge of consistency in the context of partial-order relations for ordinal variables.

This paper is not interested in adjudicating on the desirability of consistency in distributional comparisons with ordinal variables. We are content to point out that there may be good reasons for or against it depending on the context.[2] Rather, our starting point is to reflect on the differences and similarities between studying consistency of incomplete relations on ordinal variables (typically social welfare and inequality partial orderings) on the one hand, and on the other hand, consistency of inequality and polarisation indices (which previous research has already addressed, including Yalonetzky (2022) for ordinal variables). Specifically, the issue of incomparability does not arise with complete relations by definition. Therefore an index is either consistent, if for all pairs of distributions (on which the relation is defined) the ordering is preserved by reversing the order of the categories; or inconsistent otherwise. By contrast, the presence of incomparability in incomplete relations demands a more nuanced approach to consistency.

[1] See, for instance, Micklewright and Stewart (1999); Clarke et al. (2002); Kenny (2004); Erreygers (2009); Lambert and Zheng (2011); Lasso de la Vega and Aristondo (2012); Aristondo and Lasso de la Vega (2013); Silber (2015); Chakravarty et al. (2015); Kjellsson et al. (2015); Bosmans (2016); Permanyer (2016); Permanyer et al. (2022).

[2] For some interesting ethical implications consider Lambert and Zheng (2011) in the context of bounded variables.

Hence, we propose three consistency criteria. The first of these is the well known property of consistency, suitably defined in the context of incomplete relations. Namely, an incomplete relation is consistent if, for every pair of ordered distributions where $\mathbf{x}$ dominates $\mathbf{y}$, it is also the case that $\mathbf{Rx}$ dominates $\mathbf{Ry}$, such that $\mathbf{Ry}$ ensues from $\mathbf{y}$ by reversing the order of socioeconomic categories (and respectively the same for $\mathbf{x}$). Next we also distinguish situations whereby a distribution $\mathbf{x}$ dominating $\mathbf{y}$ results in $\mathbf{Ry}$ dominating $\mathbf{Rx}$. This second property, which we call anti-consistency, may arise in relations featuring anonymous Pareto improvements, such as the well-known first order stochastic dominance relation, as we show. Finally, there may be relations whereby (i) $\mathbf{x}_1$ dominates $\mathbf{y}_1$ and $\mathbf{x}_2$ dominates $\mathbf{y}_2$, but $\mathbf{Ry}_1$ dominates $\mathbf{Rx}_1$ while $\mathbf{Rx}_2$ dominates $\mathbf{Ry}_2$; or (ii) $\mathbf{x}$ dominates $\mathbf{y}$ but $\mathbf{Rx}$ and $\mathbf{Ry}$ become incomparable. This third case (including both situations (i) and (ii)) is how we define inconsistency of a partial ordering.

Our approach to consistency may be viewed as an inquiry into a particular type of isomorphism between pairs of order relations. Specifically, two order relations are isomorphic when there exists a bijective map f such that all pairs $\mathbf{x}$ and $\mathbf{y}$ in one relation are ordered in the same way as the second relation orders $f(\mathbf{x})$ and $f(\mathbf{y})$. In the context of consistency, the bijective function f is the mapping that transforms $\mathbf{x}$ into a new distribution $f(\mathbf{x})$ where the order of the socioeconomic categories is reversed.

Clearly, the reversal of categories involved in the consistency literature, is a specific type of permutation of the socioeconomic categories. In the particular case of partial welfare and inequality orders defined on convex cones (see Magdalou, 2021; Abul Naga, 2022), which we call *linear* partial orders, we can associate the three consistency properties to the nature of *the set of transfers*, namely the set of distributional transformations deemed to improve social welfare. Then we can establish the consistency, anti-consistency or inconsistency of a relation defined on convex cones by examining how its set of transfers is altered by specific types of categorical permutations. We note that these results apply to any arbitrary permutation of categories, even though our focus is on the reversal of categories. Using the results to test for alternative forms of consistency in linear partial orderings, we show, first, that the Hammond inequality partial ordering (Gravel et al., 2021) is consistent. Then we show that linear social welfare partial orderings based on anonymous Pareto improvements ('increments') are either anti-consistent, as with first-order stochastic dominance, or inconsistent as in the case of the two Hammond welfare partial orderings (Gravel et al., 2021) (one defined on 'increments' and the other on 'decrements').

Not all partial orderings pertaining to welfare or inequality with ordinal variables are defined on convex cones. Therefore, instead of relying on the set of transfers, we probe these *non-linear* relations applying directly the notions of isomorphism. Specifically we conclude that a non-linear partial order relation is consistent whenever its implementation condition[3] for

[3]Incomplete partial relations such as those considered in this paper are usually described by equivalence theorems comprising an axiomatic condition, an implementation condition and, often, a transformation condition. The axiomatic condition normally states that all ordering criteria satisfying a set of axioms (e.g., monotonicity) agree in ranking a pair of distributions. The transformation condition states that within that same pair one distribution can be obtained from the other through a finite sequence of distributional transformations (e.g., different types of transfers). Finally, the implementation condition establishes a set of

a pair of distributions where $\mathbf{x}$ dominates $\mathbf{y}$ implies, and is implied by, the implementation condition for the same relation and the corresponding pair where $f(\mathbf{x})$ dominates $f(\mathbf{y})$. Here f is the mapping that transforms $\mathbf{x}$ into a new distribution $f(\mathbf{x})$ with the order of categories reversed. With this test we show that both the median-preserving spread (Allison and Foster, 2004; Kobus, 2015) and the bipolarisation (Chakravarty and Maharaj, 2015) partial orderings are consistent, whereas the status Lorenz partial ordering (Jenkins, 2021) is inconsistent according to our proposed definitions.

The significance of our results is as follows. In empirical work involving distributional analysis on ordinal variables, it is important for the researcher to be aware of, and investigate, the effect of reversing the order of socioeconomic categories on their conclusions. The results in this paper provide analytical criteria for discriminating between consistent partial orders, which are robust to the reverse ordering of categories; anti-consistent partial orders; and those relations that are inconsistent, which demand more in-depth analysis from the practitioner. Therefore, rather than advocating for the fulfillment of a particular consistency-related property, this paper offers results that are useful to identify those partial orderings which comply with whichever normative decision one reaches regarding consistency.

The rest of the paper proceeds as follows. Section 2 provides the general setting and then proposes the notions of consistency, anti-consistency and inconsistency for incomplete relations with ordinal variables. Section 3 introduces the concept of linear partial orderings for welfare and inequality comparisons with ordinal variables and establishes the key general results pertaining to the robustness of distributional comparisons to specific types of permutations and reversals of distributional categories, followed by the specific consistency and anti-consistency tests for linear partial orderings. Its subsection 3.1 deploys these results to study first-order stochastic dominance for ordinal variables (Yalonetzky, 2013; Gravel et al., 2021), the two Hammond social welfare partial orderings and the Hammond inequality partial ordering (Gravel et al., 2021). Section 4 switches the attention to non-linear partial orderings and establishes the consistency (or lack thereof) of three inequality partial orderings prominent in the literature: median-preserving spreads (Allison and Foster, 2004; Kobus, 2015), bipolarisation (Chakravarty and Maharaj, 2015) and status Lorenz (Jenkins, 2021). The paper concludes with some remarks in section 5.

2. Three notions of consistency for incomplete partial orderings with ordinal variables

Let the sets $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the integers, rational and real numbers, respectively. Additionally, let $\mathbb{Z}_+ := \{0, 1, 2, ...\}$ denote the non-negative integers, and define the sets $\mathbb{Q}_+$ and $\mathbb{R}_+$ analogously. Let $\mathbf{u}'$ denote the transpose of a vector $\mathbf{u} \in \mathbb{R}^k$, and let $\mathbf{0}_m$ denote a vector of m zeroes.

Let there be $k > 1$ ordered socioeconomic categories from worst to best. Now, let $\mathbf{x}$ denote

tests based on comparisons of distributional features (e.g., measures of central tendency, functions of relative frequencies, etc.) which are necessary and sufficient to uphold the other statements in the theorem and the pairwise ordering itself.

a vector in $\mathbb{R}^k$, such that for each of its elements $i = 1, ..., k$, $0 \leq x_i \leq n$; and $\sum_{i=1}^{k} = n$ where $n > 0$ is a real number.[4] Then, considering subsets $\mathcal{L}$ and $\mathcal{M}$ of $\mathbb{R}^k$ and denoting a partial-order relation on that subset as $\succeq_L$, we can define an isomorphism in 1:

Definition 1 *Two partially ordered sets $(\mathcal{L}, \succeq_L)$ and $(\mathcal{M}, \succeq_M)$ are said to be order-isomorphic if there is a bijective map $f : \mathcal{L} \longrightarrow \mathcal{M}$ such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{L}$, $\mathbf{x} \succeq_L \mathbf{y}$ if and only if $f(\mathbf{x}) \succeq_M f(\mathbf{y})$.*

That is, if $(\mathcal{L}, \succeq_L)$ and $(\mathcal{M}, \succeq_M)$ are isomorphic, then $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{L}$ are ordered by $\succeq_L$ in the same way as $f(\mathbf{x})$ and $f(\mathbf{y})$ in $\mathcal{M}$ are ordered by $\succeq_M$. Moreover, $\mathbf{x}$ and $\mathbf{y}$ are incomparable in $\succeq_L$ if and only if $f(\mathbf{x})$ and $f(\mathbf{y})$ are incomparable in $\succeq_M$ when $\succeq_L$ and $\succeq_M$ are isomorphic.

Now, let $\mathbf{\Pi}$ denote the set of $k \times k$ permutation matrices and consider the reversal matrix (Horn and Johnson, 2013, p. 33) $\mathbf{R} \in \mathbf{\Pi}$:

$$\mathbf{R} := \begin{pmatrix} 0 & & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & 1 & 0 & \\ \vdots & & & & \\ 1 & 0 & \cdots & & 0 \end{pmatrix}$$

such that $\mathbf{Rz} = (z_k, z_{k-1}, .., z_1)'$. Then, if $\mathcal{S}$ denotes a set of distributions in $\mathbb{R}^k$, we can define the set of reversed distributions

$$\mathbf{R}\mathcal{S} = \{(x_k, ..., x_1) : (x_1, ..., x_k) \in \mathcal{S}\},$$

and a partial order $(\mathbf{R}\mathcal{S}, \succeq_S)$.

Then we propose a general definition of consistency, anti-consistency and inconsistency in terms of a reversing bijective mapping $f(\mathbf{x}) = \mathbf{Rx}$:

Definition 2 *A partial-order relation $(\succeq_S)$ for ordinal variables is said to be: (1) consistent if, for every ordered pair $\mathbf{x}$ and $\mathbf{y}$ in the ground set $\mathcal{S}$, $\mathbf{x} \succeq_S \mathbf{y}$ if and only if $\mathbf{Rx} \succeq_S \mathbf{Ry}$ in $\mathbf{R}\mathcal{S}$; (2) anti-consistent if, for every ordered pair $\mathbf{x}$ and $\mathbf{y}$ in the ground set $\mathcal{S}$, $\mathbf{x} \succeq_S \mathbf{y}$ if and only if $\mathbf{Ry} \succeq_S \mathbf{Rx}$ in $\mathbf{R}\mathcal{S}$; or (3) inconsistent if either:*

- *There exist at least two ordered pairs, $\mathbf{x}_1, \mathbf{y}_1$ and $\mathbf{x}_2, \mathbf{y}_2$, in $\mathcal{S}$ such that $\mathbf{x}_1 \succeq_S \mathbf{y}_1$ and $\mathbf{x}_2 \succeq_S \mathbf{y}_2$,* but *$\mathbf{Rx}_1 \succeq_S \mathbf{Ry}_1$ and $\mathbf{Ry}_2 \succeq_S \mathbf{Rx}_2$ in $\mathbf{R}\mathcal{S}$;* or
- *there exists one ordered pair $\mathbf{x}, \mathbf{y}$ such that $\mathbf{x} \succeq_S \mathbf{y}$* but *$\mathbf{Rx}$ and $\mathbf{Ry}$ are incomparable by the relation $\succeq_S$.*

[4]If $n = 1$ then $\mathbf{x}$ is a relative frequency distribution.

That is, any partial-order relation $\succeq_S$ is consistent if, for all ordered pairs $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{S}$, $\mathbf{x} \succeq_S \mathbf{y}$ if and only if $\mathbf{Rx} \succeq_S \mathbf{Ry}$ in $\mathbf{R}\mathcal{S}$. We can also envision an alternative scenario for some partial-order relation $\succeq_S$ whereby for all ordered pairs $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{S}$, $\mathbf{x} \succeq_S \mathbf{y}$ if and only if $\mathbf{Ry} \succeq_S \mathbf{Rx}$ in $\mathbf{R}\mathcal{S}$. We name this property anti-consistency.

In order to distinguish specifically inconsistent relations from relations which generally do not satisfy the consistency property (e.g., anti-consistent relations), we refer to the latter as non-consistent. Likewise, any relation violating anti-consistency is named non-anti-consistent. Naturally, the latter include both consistent and inconsistent cases.

In section 3, we focus on the class of linear social welfare and inequality orderings. These can be defined as relations satisfying a system of inequalities of the form $\mathbf{A}(\mathbf{x} - \mathbf{y}) \leq \mathbf{0}_m$ where each element of $\mathbf{A}$ is independent of $\mathbf{x}$ and $\mathbf{y}$, or equivalently via a set of vectors $\mathcal{T} := \{\mathbf{t^1}, \cdots, \mathbf{t^q}\}$ that provide the directions of increase in social welfare; the so-called set of transfers (see Magdalou, 2021). In section 3 the focus will be on how the bijective mapping $f(\mathbf{x}) = \mathbf{Rx}$ transforms the set of transfers $\mathcal{T}$ in the partially ordered set $(\mathcal{L}, \succeq_L)$ into a new set of transfers $f(\mathcal{T})$ in $(\mathcal{M}, \succeq_M)$. In section 4, however, non-linear social welfare and inequality partial orderings are more easily treated by examining how the bijective mapping f transforms the implementation criteria when going from $\mathcal{L}$ to $\mathcal{M}$.

3. Linear social welfare and inequality partial orderings for ordinal variables

We begin this section by introducing a class of order relations in definition 3, which we call *linear*. Then we review some properties of the solution set of a system of linear inequalities, and the effect of permuting some variables thereon. We need these properties in order to formulate the main results of this section.

Definition 3 *Linear order relation: Let $\mathbf{x}$ and $\mathbf{y}$ denote two vectors in $\mathbb{R}^k$, and $\mathbf{z} := \mathbf{x} - \mathbf{y}$. An order relation $\succeq$ on $\mathbb{R}^k$ is said to be linear if there exists an $m \times k$ real full-rank matrix $\mathbf{A}$ whose elements are all independent of any pair $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, such that for all distributions $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, $\mathbf{x} \succeq \mathbf{y}$ if and only if $\mathbf{Az} \leq \mathbf{0}_m$.*

Some notable examples of linear order relations are considered in subsection 3.1.

To understand how permutation of variables affects the solution of a linear system of inequalities $\mathbf{Az} \leq \mathbf{0}_m$, consider first an easier setup, that of a homogeneous linear system of *equalities* $\mathbf{Az} = \mathbf{0}_m$. Then, if $\lambda_1, \lambda_2 \in \mathbb{R}$ are any scalars, and $\mathbf{t^1}$ and $\mathbf{t^2}$ are any k-dimensional vectors such that $\mathbf{At^1} = \mathbf{0}_m$ and $\mathbf{At^2} = \mathbf{0}_m$, it is also the case that $\mathbf{s} := \lambda_1\mathbf{t^1} + \lambda_2\mathbf{t^2}$ is a solution of the homogenous system $\mathbf{Az} = \mathbf{0}_m$. That is, the solution set of the homogeneous linear system of equalities defines an algebraic structure known as a subspace. Furthermore, if r denotes the rank of $\mathbf{A}$, it follows that the solution set of this system has dimension $k - r$, meaning that any vector $\mathbf{s}$ such that $\mathbf{As} = \mathbf{0}_m$ can be expressed as an arbitrary linear combination of at most $k - r$ linearly independent vectors.[5]

[5] See Abadir and Magnus (2005, chapter 6) for further details.

In turn, in the case of a system of *inequalities* $\mathbf{Az} \leq \mathbf{0}_k$ underlying the class of linear partial orderings, if $\mathbf{t^1}$ and $\mathbf{t^2}$ are any k-dimensional vectors such that $\mathbf{At^1} \leq \mathbf{0}_m$ and $\mathbf{At^2} \leq \mathbf{0}_m$, then $\mathbf{s} := \lambda_1 \mathbf{t^1} + \lambda_2 \mathbf{t^2}$ is a solution of the system $\mathbf{Az} \leq \mathbf{0}_m$ if λ_1, λ_2 are non-negative scalars. Thus, the solution set of the linear system of inequalities is a *finitely generated cone* $\mathcal{C} := \{\lambda_1 \mathbf{t^1} + \cdots + \lambda_q \mathbf{t^q} : \lambda_1, \ldots, \lambda_q \in \mathbb{R}_+\} \subseteq \mathbb{R}^k$. Moreover, save for exceptional cases, it is generally not possible to determine q (the cardinality of the set of solutions) as a function of k and the rank of the matrix r.

Let $\mathbf{\Pi}$ denote the set of permutation matrices in $\mathbb{R}^k$. We note that if $\mathbf{P} \in \mathbf{\Pi}$ is a permutation matrix, then $\mathbf{P}$ is an orthogonal matrix. That is, $\mathbf{P}$ is an invertible matrix and the inverse $\mathbf{P^{-1}}$ equals the transpose of $\mathbf{P}$, namely $\mathbf{P'}$ which is also a permutation matrix. From here on, we consider two systems of inequalities:

$$\mathbf{Az} \leq \mathbf{0}_m, \tag{1}$$

and for some permutation matrix $\mathbf{P} \in \mathbf{\Pi}$

$$\mathbf{APz} \leq \mathbf{0}_m. \tag{2}$$

We call (1) the *principal system*, and (2) the *auxiliary system*. Because the solution set to a system of m linear inequalities in k variables is a finitely generated cone, we can define a set $\mathcal{T} \subseteq \mathbb{R}^k$,

$$\mathcal{T} := \left\{\mathbf{t^1}, \cdots, \mathbf{t^q}\right\}$$

such that

$$\begin{aligned} \left\{\mathbf{z} \in \mathbb{R}^k : \mathbf{Az} \leq \mathbf{0}_m\right\} &= \\ &= \left\{\lambda_1 \mathbf{t^1} + \cdots + \lambda_q \mathbf{t^q} : \lambda_1, \ldots, \lambda_q \geq 0\right\} = \text{cone}(\mathcal{T}). \end{aligned}$$

Cone($\mathcal{T}$) is said to be a pointed cone if for all $\mathbf{s}$ elements of cone($\mathcal{T}$) $\setminus \mathbf{0}_k$, $-\mathbf{s}$ is not an element of cone($\mathcal{T}$). If the matrix $\mathbf{A}$ is of full rank, then cone($\mathcal{T}$) is pointed (Burns et al., 1974). Following Magdalou (2021), we may call $\mathcal{T}$ *the set of transfers*. The first question we address establishes the relation between the solution sets of the principal and auxiliary system. For this purpose, we let $\mathcal{T}^* := \{\mathbf{t^{*1}}, \cdots, \mathbf{t^{*r}}\}$ denote the solution set of the auxiliary system (2), such that

$$\begin{aligned} \left\{\mathbf{z} \in \mathbb{R}^k : \mathbf{APz} \leq \mathbf{0}_m\right\} &= \\ &= \left\{\delta_1 \mathbf{t^{*1}} + \cdots + \delta_r \mathbf{t^{*r}} : \delta_1, \ldots, \delta_r \geq 0\right\} = \text{cone}(\mathcal{T}^*). \end{aligned}$$

Finally gather in a $k \times q$ matrix $\mathbf{T}$ the vectors $\mathbf{t^1}, \cdots, \mathbf{t^q}$, and in a $k \times r$ matrix $\mathbf{T}^*$ the vectors $\mathbf{t^{*1}}, \cdots, \mathbf{t^{*r}}$. Now consider lemmas 3.1 and 3.2 which explore relations between the solution sets of $\mathbf{Az} \leq \mathbf{0}_m$ and $\mathbf{APz} \leq \mathbf{0}_m$:

Lemma 3.1 *Let $\mathbf{P} \in \mathbf{\Pi}$ denote a $k \times k$ permutation matrix, and let $\mathcal{T} := \{\mathbf{t^1}, \cdots, \mathbf{t^q}\} \subseteq \mathbb{R}^k$ denote a set of transfers. Then $\{\mathbf{z} : \mathbf{Az} \leq \mathbf{0}_m\} =$ cone($\mathcal{T}$) if and only if $\{\mathbf{z} : \mathbf{APz} \leq \mathbf{0}_m\} =$ cone($\mathbf{P'}\mathbf{t^1}, \cdots, \mathbf{P'}\mathbf{t^q}$).*

Proof

$$\begin{aligned}
\{\mathbf{z} : \mathbf{Az} \leq \mathbf{0}_m\} &= \text{cone}(\mathcal{T}) \Longleftrightarrow \\
\mathbf{AT} &\leq \mathbf{0}_{m\times q} \Longleftrightarrow \\
\mathbf{APP'T} &\leq \mathbf{0}_{m\times q} \Longleftrightarrow \\
\{\mathbf{z} : \mathbf{APz} \leq \mathbf{0}_m\} &= \text{cone}\{\mathbf{P't^1}, \cdots, \mathbf{P't^q}\}
\end{aligned}$$

where the logical equivalence between the second and the third line of the proofs follows from the fact that any permutation matrix is orthogonal. ■

Crucially, lemma 3.1 establishes a unique direct correspondence between the respective solution sets of $\mathbf{Az} \leq \mathbf{0}_m$ and $\mathbf{APz} \leq \mathbf{0}_m$. When $\mathbf{z}$ is the difference between two distributions, lemma 3.1 states that a particular set of transfers (linearly combined with positive coefficients in myriad ways) satisfies $\mathbf{Az} \leq \mathbf{0}_m$ (which in turn relates to an order relation between the distributions in $\mathbf{z}$) if and only if permuting each transfer in the set by the same permutation matrix $\mathbf{P}$ yields a set of transfers satisfying $\mathbf{APz} \leq \mathbf{0}_m$. Thus lemma 3.1 enables us to write the solution set of $\mathbf{APz} \leq \mathbf{0}_m$ as a function of the solution set of $\mathbf{Az} \leq \mathbf{0}_m$ and $\mathbf{P}$. Then we can use this lemma to prove in lemma 3.2 that, in fact, the solution sets of both principal and auxiliary systems are identical if and only if the associated sets of transfers are the same. Put differently, the solution sets of both principal and auxiliary system are identical if and only if each respective set of transfers is a collection of distinct pairs of transfers, comprising a given transfer $\mathbf{t}$ and its permuted counterpart $\mathbf{P't}$:

Lemma 3.2 *Let* $\mathbf{P} \in \mathbf{\Pi}$ *denote a* $k \times k$ *permutation matrix. Then* $\{\mathbf{z} : \mathbf{Az} \leq \mathbf{0}_m\} = \{\mathbf{z} : \mathbf{APz} \leq \mathbf{0}_m\}$ *if and only if*

(i) for all $\mathbf{t^i} \in \mathcal{T}$*, there exists* $\mathbf{t^{*j}} \in \mathcal{T}^*$ *such that* $\mathbf{t^i} = \mathbf{t^{*j}}$*, and*

(ii) for all $\mathbf{t^{*g}} \in \mathcal{T}^*$*, there exists* $\mathbf{t^h} \in \mathcal{T}$ *such that* $\mathbf{t^{*g}} = \mathbf{t^h}$*.*

Proof From Lemma 3.1, the solution sets of the principal and auxiliary systems are given by

$$\{\mathbf{z} : \mathbf{Az} \leq \mathbf{0}_m\} = \left\{\lambda_1 \mathbf{t^1} + \cdots + \lambda_q \mathbf{t^q} : \lambda_1, ..., \lambda_q \geq 0\right\} \tag{3}$$
$$\{\mathbf{z} : \mathbf{APz} \leq \mathbf{0}_m\} = \left\{\mu_1 \mathbf{t^{*1}} + \cdots + \mu_q \mathbf{t^{*q}} : \mu_1, ..., \mu_q \geq 0\right\} \tag{4}$$

Therefore, the solution set of the principal system (3) is a subset of the solution set of (4), if $\mathcal{T} \subseteq \mathcal{T}^*$; equivalently if condition (i) holds. Conversely, the solution set of the auxiliary system is a subset of the solution set of the principal system , if $\mathcal{T}^* \subseteq \mathcal{T}$; equivalently if condition (ii) holds. Finally, the principal and auxiliary systems have the same solution set if and only if $\mathcal{T}^* = \mathcal{T}$; equivalently, if and only if (i) and (ii) hold. ■

Now we are ready to apply lemmas 3.1 and 3.2 to our stated problem. First, note that the reversal matrix $\mathbf{R}$ is symmetric. Furthermore, given that $\mathbf{R}$ is an orthogonal matrix, we have $\mathbf{R}' = \mathbf{R} = \mathbf{R}^{-\mathbf{1}}$. Then, proposition 3.1 shows that a partial ordering is consistent between the two alternative categorical sorting methods if and only if reversing all the elements within the set of transfers yields transfers which also belong in the same set:

Proposition 3.1 *A linear partial ordering defined by the system of inequalities* $\mathbf{Az} \leq \mathbf{0}_m$ *is consistent if and only if for all* $\mathbf{s} = (s_1, ..., s_k)' \in \mathcal{T}$*, the vector* $\mathbf{s}^* := \mathbf{Rs} = (s_k, ..., s_1)'$ *belongs in the set of transfers* $\mathcal{T}$*.*

Proof We first recall that $\mathbf{R}$ is a symmetric and orthogonal matrix. The result then follows as a consequence of lemma 3.2 (which in turn depends on lemma 3.1). ■

Analogously, proposition 3.2 shows that a partial ordering is anti-consistent if and only if reversing all the elements within the set of transfers, and then multiplying them by minus one, yields transfers which also belong in the same set:

Proposition 3.2 *A linear partial ordering defined by the system of inequalities* $\mathbf{Az} \leq \mathbf{0}_m$ *is anti-consistent if and only if for all* $\mathbf{s} = (s_1, ..., s_k)' \in \mathcal{T}$*, the vector* $\mathbf{s}^* := -\mathbf{Rs} = (-s_k, ..., -s_1)'$ *belongs in the set of transfers* $\mathcal{T}$*.*

Proof Let $\mathbf{Az} = \mathbf{x} - \mathbf{y}$ and let $\mathbf{I} \in \mathbf{\Pi}$ be the k-dimensional identity matrix. Then:

$$\begin{aligned}
\{\mathbf{z} : \mathbf{Az} \leq \mathbf{0}_m\} &= \text{cone}(\mathcal{T}) \iff \\
\mathbf{AT} &\leq \mathbf{0}_{m\times q} \iff \\
\mathbf{A}(-\mathbf{I})(-\mathbf{I})\mathbf{T} &\leq \mathbf{0}_{m\times q} \iff \\
\{\mathbf{z} : \mathbf{A}(-\mathbf{I})\mathbf{z} \leq \mathbf{0}_m\} &= \text{cone}(-\mathcal{T}),
\end{aligned}$$

where $-\mathcal{T} = (-\mathbf{t^1}, ..., -\mathbf{t^q})$. Next, applying lemma 3.1 we have: $\{\mathbf{z} : -\mathbf{ARz} \leq \mathbf{0}_m\} = \text{cone}(-\mathbf{Rt^1}, ..., -\mathbf{Rt^q})$. Then, using the same reasoning as in lemma 3.2, it follows that $\{\mathbf{z} : \mathbf{Az} \leq \mathbf{0}_m\} = \{\mathbf{z} : -\mathbf{ARz} \leq \mathbf{0}_m\}$ if and only if for all $\mathbf{s} = (s_1, ..., s_k)' \in \mathcal{T}$, the vector $\mathbf{s}^* := -\mathbf{Rs} = (-s_k, ..., -s_1)'$ is also an element of the set of transfers $\mathcal{T}$.

■

To demonstrate the usefulness of propositions 3.1 and 3.2, we next investigate the consistency properties of some well-known partial order relations for distributional analysis on ordinal variables.

3.1. Applications

3.1.1. First-order stochastic dominance

The first-order stochastic dominance (henceforth FOD) partial ordering is linear and its matrix $\mathbf{A}$ is a lower triangular matrix of ones:

$$\mathbf{A_{k\times k}} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 1 & 1 & \cdots & & 1 \end{pmatrix}$$

Let $\mathbf{t} = (t_1, ..., t_k)$, where for some $i < k$ $t_i = -1$, $t_{i+1} = 1$ and $t_j = 0$ for all $j \neq i, i+1$. We can construct $k-1$ such vectors and, following Gravel et al. (2021), we call them increments. Then, $\mathbf{x} \succeq_{FOD} \mathbf{y}$ ($\mathbf{x}$ first-order dominates $\mathbf{y}$) if and only if $\mathbf{A}(\mathbf{x}-\mathbf{y}) \leq \mathbf{0}_k$, which in turn is equivalent with being able to obtain $\mathbf{x}$ from $\mathbf{y}$ through a sequence of increments.

We can test the consistency of the FOD partial ordering with proposition 3.1 by noting first that the set of transfers is comprised of the set of $k-1$ Pareto improvements, or increments (Gravel et al., 2021). For instance, when $k = 4$ the vector $\mathbf{z} = (0, -1, 1, 0)'$ is an increment (i.e., one person moving from the second worst category to the second best). Then $\mathbf{Rz} = (0, 1, -1, 0)'$ becomes a typical decrement (Gravel et al., 2021), namely the exact opposite of a Pareto improvement, which is not in the set of transfers corresponding to FOD. Hence we conclude that FOD is not consistent. Moreover, if $\mathbf{z} = \mathbf{x} - \mathbf{y}$ then, in the previous example, we can easily deduce that $\mathbf{x} \succeq_{FOD} \mathbf{y}$ and $\mathbf{Ry} \succeq_{FOD} \mathbf{Rx}$ because we can always undo the decrement to obtain $\mathbf{Ry}$ back from $\mathbf{Rx}$ through the corresponding reverse decrement, i.e., an increment. In fact, using proposition 3.2, we note that the negative of a reverse increment (i.e., a decrement) is also an increment; that is, $\mathbf{t}$ is an increment if and only if $-\mathbf{Rt}$ is an increment. Hence we also conclude that FOD is anti-consistent. Corollary 3.1 collects these equivalent results:

Corollary 3.1 *The following statements are equivalent: For some natural number $k > 1$, reversal matrix $\mathbf{R}$ and any pair of frequency distributions $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$:*

1. $\mathbf{x} \succeq_{FOD} \mathbf{y}$.
2. $\mathbf{Ry} \succeq_{FOD} \mathbf{Rx}$.
3. *$\mathbf{x}$ can be obtained from $\mathbf{y}$ through a finite sequence of increments; likewise $\mathbf{Ry}$ can be obtained from $\mathbf{Rx}$ through a finite sequence of increments.*
4. *$\mathbf{Rx}$ can be obtained from $\mathbf{Ry}$ through a finite sequence of decrements.*

Proof: Direct application of propositions 3.1 and 3.2 combined with the equivalence theorem of first-order dominance (e.g. see Gravel et al., 2021, theorem 1). ■

3.1.2. The Hammond inequality ordering

Gravel et al. (2021) introduced the concept of a Hammond progressive transfer for ordinal variables, whereby an individual moves from category i to j jointly with another one who moves from q to p such that $1 \leq i < j \leq p < q \leq k$. That is, a Hammond transfer renders a pair of people or units closer to each other. The Hammond inequality partial ordering, whose relation we denote by $\succeq_{HI}$, is linear (Gravel et al., 2021, theorem 5). Therefore we can test its consistency using proposition 3.1. The key property is that the reversal of any Hammond transfer remains a Hammond transfer. For instance, when $k = 6$, $\mathbf{z} = (0, -1, 0, 2, 0, -1)'$ is a Hammond transfer. Then, its reverse $\mathbf{Rz} = (-1, 0, 2, 0, -1, 0)'$ is also a Hammond transfer. Same, for instance, with $\mathbf{z} = (-1, 1, 0, 1, 0, -1)'$, which becomes $\mathbf{Rz} = (-1, 0, 1, 0, 1, -1)'$ when reversed, and so forth. Therefore, by way of corollary to proposition 3.1 we conclude that *the Hammond inequality partial ordering is consistent*:

Corollary 3.2 *For some natural number $k > 1$ and reversal matrix $\mathbf{R}$ and any pair of frequency distributions $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$, $\mathbf{x} \succeq_{HI} \mathbf{y}$ if and only if $\mathbf{Rx} \succeq_{HI} \mathbf{Ry}$.*

Proof: Direct application of proposition 3.1. ■

3.1.3. The Hammond social welfare ordering

Gravel et al. (2021) also introduced the Hammond social welfare partial ordering for ordinal variables. Based on their equivalence theorem, we say that $\mathbf{x} \succeq_{H+} \mathbf{y}$, namely $\mathbf{x}$ is preferable to $\mathbf{y}$ in terms of Hammond social welfare, if and only if $\mathbf{x}$ is obtained from $\mathbf{y}$ through a finite sequence of increments and/or Hammond transfers. That is, the Hammond social welfare partial ordering reflects concerns for both Pareto improvement and Hammond egalitarianism. This partial ordering is linear and its matrix $\mathbf{A}$ is a lower triangular matrix with typical positive entries $a_{ij} = 2^{i-j}$ for $i \geq j$ and $a_{ij} = 0$ otherwise, with $i = 1, ..., k-1$ (see Gravel et al., 2021). Therefore, we can use proposition 3.1 to test the consistency of this partial ordering. We know already from subsections 3.1.1 and 3.1.2 that reversed Hammond transfers remain Hammond transfers whereas reversed increments become decrements. Then, since the Hammond social welfare partial ordering relies on both types of transfer, we must conclude from proposition 3.1 that the Hammond social welfare partial ordering is not consistent.

Additionally, we can use proposition 3.2 to test its anti-consistency. We know from section 3.1.1 that multiplying a reversed increment by minus one yields an increment. However, multiplying a reversed Hammond transfer by minus one yields a spread whereby the two units involved end up further apart (think about the regressive counterpart to the Hammond transfer). Hence, this new transfer does not belong in the set associated with the Hammond social welfare partial ordering. Therefore, the latter cannot be anti-consistent either. In fact, the Hammond social welfare partial ordering is inconsistent:

Corollary 3.3 *The Hammond social welfare ordering is inconsistent.*

Proof: Considering $\mathbf{x}$, $\mathbf{y}$, $\mathbf{r}$ and $\mathbf{w}$ in $\mathbb{R}^k$; let $\mathbf{x} \succeq_{H+} \mathbf{y}$ and $\mathbf{r} \succeq_{H+} \mathbf{w}$, but $\mathbf{x}$ is obtained from $\mathbf{y}$ exclusively through a sequence of increments, whereas $\mathbf{r}$ is obtained from $\mathbf{w}$ exclusively through a sequence of Hammond transfers. Then we can show that $\mathbf{Ry} \succeq_{H+} \mathbf{Rx}$ and $\mathbf{Rr} \succeq_{H+} \mathbf{Rw}$. This fits precisely the description of the first inconsistency scenario in definition 2. ■

3.2. Reverse iso-morphism between pairs of order relations

We finish the discussion of consistency among linear partial orderings, noting that other iso-morphisms involving reversal matrices can be deduced, even when the partial orderings are inconsistent. Take the case of the two Hammond welfare partial orderings. In addition to the partial order relation $\succeq_{H+}$, Gravel et al. (2021) replaced increments with decrements in order to define another relation whereby $x \succeq_{H-} y$, namely $\mathbf{x}$ is preferable to $\mathbf{y}$ in terms of Hammond social welfare with decrements, if and only if $\mathbf{x}$ is obtained from $\mathbf{y}$ through a finite sequence of decrements and/or Hammond transfers. Think about social bads (e.g. pollution) as an empirical motivation for this type of ordering. This partial ordering is also linear and its matrix A is an upper triangular matrix with typical positive entries $a_{ij} = 2^{j-i-1}$ for $i \leq j$ and $a_{ij} = 0$ otherwise, with $i = 1, ..., k-1$ (see Gravel et al., 2021).

The pair of relations $\succeq_{H+}$ and $\succeq_{H-}$ belong to a class which we define as follows:

Definition 4 *Two partially ordered sets $(\mathcal{S}, \succeq_L)$ and $(\mathcal{S}, \succeq_M)$ are said to be reverse order-isomorphic, if for the bijective map $f : \mathcal{S} \longrightarrow \mathbf{R}\mathcal{S}$, and for all $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, $\mathbf{x} \succeq_L \mathbf{y}$ if and only if $\mathbf{Rx} \succeq_M \mathbf{Ry}$.*

Then we can establish a reverse iso-morphism between the pair of relations $\succeq_{H+}$ and $\succeq_{H-}$, as stated in corollary 3.4:

Corollary 3.4 *The following statements are equivalent: For some natural number $k > 1$, reversal matrix $\mathbf{R}$ and any pair of frequency distributions $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$:*

1. $\mathbf{x} \succeq_{H+} \mathbf{y}$.
2. $\mathbf{Rx} \succeq_{H-} \mathbf{Ry}$.
3. *$\mathbf{x}$ can be obtained from $\mathbf{y}$ through a finite sequence of increments and/or Hammond transfers.*
4. *$\mathbf{Rx}$ can be obtained from $\mathbf{Ry}$ through a finite sequence of decrements and/or Hammond transfers.*

Proof: For proving that (3) if and only if (4), recall that reversing an increment yields a decrement (and vice-versa), whereas reversing a Hammond transfer yields another Hammond transfer (corollary 3.2). Then the two equivalence theorems of Hammond social welfare

(Gravel et al., 2021, theorems 3 and 4) prove the equivalences between statement pairs (1)-(3) and (2)-(4), respectively. ■

Finally, note that a similar reverse isomorphism can be identified between first-order dominance and a counterpart welfare criterion defined solely over decrements.

4. Non-linear social welfare and inequality partial orderings for ordinal variables

In the following subsections, we probe the consistency (or lack thereof) of prominent partial orderings for ordinal variables including (i) median-preserving spreads, (ii) bipolarisation, and (iii) the status Lorenz criterion. Their non-linearity precludes the application of propositions 3.1 and 3.2. However, let $(\mathcal{S}, \succeq_S)$ denote a partial order and $(\mathbf{R}\mathcal{S}, \succeq_S)$ denote the relation after reversing the order of categories. Returning to definition 1, note that the function $f : \mathcal{S} \to \mathbf{R}\mathcal{S}$ given by $f(\mathbf{x}) = \mathbf{R}\mathbf{x}$ is bijective. Therefore, in the discussion below consistency is probed by examining whether for all $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, $\mathbf{x} \succeq_S \mathbf{y}$ if and only if $\mathbf{R}\mathbf{x} \succeq_S \mathbf{R}\mathbf{y}$, which is the very definition of the property. Note that a key straightforward implication of this definition (2) is the mutual implication between the implementation condition of every pair $\mathbf{x}, \mathbf{y}$ in $\mathcal{S}$ ordered by relation $\succeq_S$ and the implementation condition of its counterpart $\mathbf{R}\mathbf{x}, \mathbf{R}\mathbf{y}$ in $\mathbf{R}\mathcal{S}$ ordered by $\succeq_S$, when the partial-order relation $(\mathcal{S}, \succeq_S)$ is consistent. Thus, for each aforementioned non-linear partial ordering we test for consistency focusing on its implementation condition.

4.1. Median preserving spreads and bipolarisation

Several inequality measures are sensitive to so-called median-preserving spreads (Mendelson, 1987; Allison and Foster, 2004; Kobus, 2015), thereby respecting the median-preserving-spread (henceforth MPS) partial ordering. MPS are movements of probability mass (e.g., people) away from the distribution's median in a way that increases dispersion and produces "fatter tails".

Consider relative frequency distributions of ordinal variables such as $\mathbf{p} \in \mathcal{O}^k$ with typical element p_i, where $\mathcal{O}^k$ is the set of all frequency distributions with $k > 1$ categories (i.e., all elements of $\mathbf{p}$ are non-negative and $\mathbf{1}_\mathbf{k}'\mathbf{p} = 1$). Likewise, $\mathbf{P}$ and $\overline{\mathbf{P}}$ stand for the cumulative distribution and survival functions of $\mathbf{P}$, respectively. Let the median, m_e, be the category ensuring $P_{m_e - 1} < 0.5$ and $P_{m_e} \geq 0.5$.[6] Also let $\mathbf{R}\mathbf{p}$ be the reversed ordered distribution corresponding to $\mathbf{p}$, with typical element $p_i^R = p_{k-i+1}$ for all $i = 1, ..., k$ (same for cumulative and survival functions) and median $m_e^R = k - m_e + 1$.

Then, let $\succeq^{MPS}$ denote the MPS partial ordering, such that $\mathbf{p} \succeq^{MPS} \mathbf{q}$ means that $\mathbf{p}$ is a median-preserving spread of $\mathbf{q}$, namely the former is obtained from the latter through a finite sequence of median-preserving spreads.

[6] For simplicity we focus on the case of a single common median, but the definitions and results can be extended to the case of several common medians.

Apouey (2007) and Chakravarty and Maharaj (2015) pioneered the measurement of bipolarisation with ordinal variables. Both bipolarisation indices and partial orderings are sensitive to median-preserving spreads, namely 'fatter tails' should correspond to higher bipolarisation. But additionally, they abide by the increasing bipolarisation property, whereby bringing two people on one side of the median closer (essentially, a Hammond transfer on one side of the median) should lead to higher bipolarisation (i.e., higher clustering on either side of the median).[7]

Let $\succeq^B$ denote the bipolarisation partial ordering. Chakravarty and Maharaj (2015, theorem 5) provide a two-statement definition of the bipolarisation partial ordering, whose implementation condition proceeds as follows:[8]

$$\mathbf{p} \succeq^B \mathbf{q} \text{ if and only if}$$

$$\text{i} \sum_{i=j}^{m_e-1} P_i \leq \sum_{i=j}^{m_e-1} Q_i \text{ for all } 1 \leq j \leq m_e - 1 \text{ and (ii) } \sum_{i=m_e}^{j} P_i \geq \sum_{i=m_e}^{j} Q_i \text{ for all } m_e \leq j \leq k-1. \quad (5)$$

Neither the bipolarisation partial ordering nor the MPS partial ordering are linear in the sense of definition 3, because even though we can write a matrix $\mathbf{A}$ for each of them, the matrix's elements are a function of the median, which in turn is a function of the elements of $\mathbf{p}$ and $\mathbf{q}$. Hence the consistency assessment for the bipolarisation partial ordering in proposition 4.1, relies on the implementation conditions test, as opposed to proposition 3.1. That is, we test the consistency of this partial ordering by checking whether the implementation condition associated with $(\mathcal{O}^k, \succeq^B)$ (for any $k > 1$) holds if and only if the implementation condition associated with $(\mathbf{R}\mathcal{O}^k, \succeq^B)$ also holds.

Proposition 4.1 *The bipolarisation partial ordering is consistent in the sense of definition 2.*

Proof:

The bipolarisation relation $\succeq^B$ orders distributions in a subset of $\mathcal{O}^k$ defined by distributions sharing median m_e. Then the mapping $\mathbf{R}$ sends the partial order $(\mathcal{O}^k_{me}, \succeq^B)$ to the subset of distributions in ($\mathcal{O}^k$ with median $k - m_e + 1$ and the relation $\succeq^B$, namely $(\mathcal{O}^k_{k-me+1}, \succeq^B)$. Because the mapping $f(\mathbf{x}) = \mathbf{R}\mathbf{x}$ is bijective, in order to show that there is an isomorphism between the two aforementioned partial orders, we just need to prove that the implementation condition for $\mathbf{p} \succeq^B \mathbf{q}$ holds if and only if the implementation condition for $\mathbf{Rp} \succeq^B \mathbf{Rq}$ holds;

[7]However, note that the Hammond inequality partial ordering neither implies nor is implied by the bipolarisation partial ordering even though the latter relies on Hammond transfers on one side of the median. Hence we cannot use corollary 3.2 stating the consistency of the Hammond inequality partial ordering to assess the bipolarisation partial ordering.

[8]For simplicity we focus on the case of a single common median, but the definitions and results can be extended to the case of several common medians.

bearing in mind the equivalence between the implementation condition in (5) and the other statements defining the bipolarisation partial order (see Chakravarty and Maharaj, 2015).

That is, we need to prove that $\sum_{i=j}^{m_e-1} P_i \leq \sum_{i=j}^{m_e-1} Q_i$ for all $1 \leq j \leq m_e - 1$ and $\sum_{i=m_e}^{j} P_i \geq \sum_{i=m_e}^{j} Q_i$ for all $m_e \leq j \leq k-1$, if and only if $\sum_{i=j}^{m_e^R-1} P_i^R \leq \sum_{i=j}^{m_e^R-1} Q_i^R$ for all $1 \leq j \leq m_e^R - 1$ and $\sum_{i=m_e^R}^{j} P_i^R \geq \sum_{i=m_e^R}^{j} Q_i^R$ for all $m_e^R \leq j \leq k-1$, where $m_e^R = k - m_e + 1$. The proof is as follows:

$\sum_{i=j}^{m_e-1} P_i \leq \sum_{i=j}^{m_e-1} Q_i$ for all $1 \leq j \leq m_e - 1$ if and only if $\sum_{i=k-m_e^R}^{j} P_{k-i}^R \geq \sum_{i=k-m_e^R}^{j} Q_{k-i}^R$ for all $k - m_e^R \leq j \leq k-1$, because $P_i = 1 - P_{k-i}^R$ (for all $i = 1, ..., k-1$) and $m_e^R = k - m_e + 1$.

$\sum_{i=m_e}^{j} P_i \geq \sum_{i=m_e}^{j} Q_i$ for all $m_e \leq j < k$ if and only if $\sum_{i=k-j}^{k-m_e^R+1} P_{k-i}^R \leq Q_{k-i}^R$ for all $1 \leq j < m_e^R$, because $P_i = 1 - P_{k-i}^R$ (for all $i = 1, ..., k-1$) and $m_e^R = k - m_e + 1$. ■

Though not stated in Chakravarty and Maharaj (2015, theorems 1 and 3), it is easy to show that $\mathbf{p} \succeq^B \mathbf{q}$ if and only if $\mathbf{p}$ can be obtained from $\mathbf{q}$ through a finite sequence of median-preserving spreads and/or clustering transfers (i.e., Hammond transfers on one side of the median). Besides being intrinsically valuable, this latter statement helps us conclude that the MPS partial ordering is also consistent by way of corollary to proposition 4.1:

Corollary 4.1 *The MPS partial ordering is consistent.*

Proof: Since $\mathbf{p} \succeq^B \mathbf{q}$ if and only if $\mathbf{p}$ can be obtained from $\mathbf{q}$ through a finite sequence of MPS and/or clustering transfers, then it is the case that $(\mathcal{O}_{me}^k, \succeq^{MPS}) \subset (\mathcal{O}_{me}^k, \succeq_{m_e}^B)$. Therefore, since the definition of consistency pertains to every ordered pair within the partial ordered set, it must be true that the consistency of the bipolarisation partial ordering implies the consistency of the MPS partial ordering. ■

4.2. Status Lorenz

Cowell and Flachaire (2017) proposed measuring inequality with ordinal variables in terms of dispersion of people's personal status, which is particularly useful for comparing distributions without common medians. They proposed operationalising personal status in four possible ways, including peer-inclusive downward-looking status in the form of the proportion of people in the same category or worse. This subsection focuses on this definition of status (but similar results hold for the other alternatives).[9]

Then, Cowell and Flachaire (2017) axiomatically characterised a single-parameter class of status inequality indices. As with other such classes, theirs admits several members depending on the choice of the single parameter. Hence, Jenkins (2021) proposed a dominance

[9] The other three definitions of status proposed by Cowell and Flachaire (2017) are: peer-exclusive downward-looking status which is the proportion of people in any worse category; peer-inclusive upward-looking status which is measured by the proportion of people in the same category or better; and peer-exclusive upward-looking status which is the proportion of people in any better category.

condition whose fulfilment guarantees the robustness of a status inequality comparison to any alternative choice of index belonging in the class proposed by Cowell and Flachaire (2017) and, generally, for any inequality index $I : [0,1]^n \to \mathbb{R}$ that is decreasing and convex (where n is the population size as every person is attributed a status equal to the proportion of people in the same category or worse). The subsequent partial ordering, $\succeq^S$, is based on the generalised Lorenz curve for ordinal variables presented in 6:

$$GL(\mathbf{P}; j) = \frac{1}{n} \sum_{i=1}^{j} P_i^*, \; j = 1, ..., n, \tag{6}$$

where, for every individual i: $P_i = \sum_{j=1}^{k} P_j \mathbb{I}(i \in \int_j)$, $\mathbb{I}(.) = 1$ if the statement in parenthesis is true (otherwise $\mathbb{I}(.) = 0$) and $\int_j$ is the set of individuals in category j. That is, each individual's status is measured by the proportion of people in the same category or worse. Finally, the asterisk in P_i^* means that the statuses are ordered from lowest to highest.

For comparisons with populations of *equal size*, the related implementation condition (Jenkins, 2021) can be expressed as follows:

$$\mathbf{p} \succeq^S \mathbf{q} \text{ if and only if } GL(\mathbf{p}; j) \geq GL(\mathbf{q}; j) \text{ for all } j = 1, ..., n. \tag{7}$$

The status Lorenz partial ordering is not linear. In fact, for any $j = 1, ..., n$, it is easy to show that this partial ordering remains a non-linear function of the frequencies in $\mathbf{p}$ (see 6). Hence this partial ordering cannot be represented by a matrix $\mathbf{A}$ according to definition 3. Therefore, proposition 3.1 is not applicable and the consistency assessment for the Lorenz status partial ordering stated in proposition 4.2 relies on the implementation condition (again):

Proposition 4.2 *The status Lorenz partial ordering is inconsistent.*

Proof: We need to prove that it is not true that, for any pair $\mathbf{p}, \mathbf{q} \in \mathcal{O}^k$, $GL(\mathbf{p}; j) \geq GL(\mathbf{q}; j)$ for all $j = 1, ..., n$ if and only if $GL(\mathbf{Rp}; j) \geq GL(\mathbf{Rq}; j)$ for all $j = 1, ..., n$. There are different ways to reach this conclusion. We use the counterexample of a pair $\mathbf{p}, \mathbf{q} \in \mathcal{O}^k$ characterised by $GL(\mathbf{p}; j) \geq GL(\mathbf{q}; j)$ for all $j = 1, ..., n$ and $GL(\mathbf{Rp}; j) < GL(\mathbf{Rq}; j)$ for some $j = 1, ..., n$.

Let $n = 10$ with $\mathbf{p} = (0.3, 0.5, 0.2)$ and $\mathbf{q} = (0.3, 0.4, 0.3)$. Then, for each $j = 1, ..., n$, the values of $GL(\mathbf{p}; j)$, $GL(\mathbf{q}; j)$, $GL(\mathbf{Rp}; j)$ and $GL(\mathbf{Rq}; j)$ appear on table 1.

Table 1 shows that $GL(\mathbf{p}; j) \geq GL(\mathbf{q}; j)$ for all $j = 1, ..., 10$; however, $GL(\mathbf{Rp}; j) < GL(\mathbf{Rq}; j)$ for $j = 1, 2$ and $GL(\mathbf{Rp}; j) \geq GL(\mathbf{Rq}; j)$ for $j = 3, ..., 10$ (the Lorenz curves of the reversed distributions 'cross'). Hence, in the case of the status Lorenz partial ordering, the mapping $\mathbf{R}$ takes us from a comparable pair ($\mathbf{p}$ and $\mathbf{q}$) to an incomparable pair ($\mathbf{Rp}$ and $\mathbf{Rq}$). As a result, the status Lorenz partial ordering is inconsistent. ■

Table 1: An inconsistent inequality comparison with status Lorenz curves

j	1	2	3	4	5	6	7	8	9	10
$GL(\mathbf{p};j)$	0.03	0.06	0.09	0.14	0.19	0.24	0.29	0.34	0.36	0.38
$GL(\mathbf{q};j)$	0.03	0.06	0.09	0.13	0.17	0.21	0.25	0.28	0.31	0.34
$GL(\mathbf{Rp};j)$	0.02	0.04	0.09	0.14	0.19	0.24	0.29	0.32	0.35	0.37
$GL(\mathbf{Rq};j)$	0.03	0.06	0.09	0.13	0.17	0.21	0.25	0.28	0.31	0.34

5. Conclusions

Arguably, if one favours (respectively opposes) consistent inequality or welfare comparisons with ordinal variables, then analytical coherence minimally demands choosing among measurement criteria which satisfy (respectively violate) the consistency desideratum. For the case of robust inequality and welfare comparisons relying on partial orderings we show that some criteria (such as Hammond inequality or bipolarisation) are consistent whereas others (such as Hammond social welfare or status Lorenz inequality) are not. Thus, our results are useful to identify those partial orderings which comply with whichever normative decision one reaches regarding consistency.

On that note, our results for linear inequality partial orderings (propositions 3.1 and 3.2) enable the axiomatisation of consistency and anti-consistency into any future proposal of linear partial orderings by imposing $\mathcal{T} = \mathbf{R}\mathcal{T}$ for consistency or $\mathcal{T} = -\mathbf{R}\mathcal{T}$ for anti-consistency, where $\mathcal{T}$ is the set of transfers defining the partial ordering and $\mathbf{R}$ is the reversal matrix.

Finally, as expected, welfare partial orders sensitive to Pareto improvements are not consistent because reversing an increment yields a decrement. We defined pairs of relations $\succeq_L$ and $\succeq_M$ to be reverse-order isomorphic whenever $\mathbf{x}$ and $\mathbf{y}$ are ordered by $\succeq_L$ if and only if the reverse distributions $\mathbf{Rx}$ and $\mathbf{Ry}$ are similarly ordered by $\succeq_M$. As such, the two relations $\succeq_{H+}$ and $\succeq_{H-}$ proposed by Gravel et al. (2021), respectively the transitive closure of increments and Hammond transfers, and the transitive closure of decrements and Hammond transfers, belong to the class of reverse-order isomorphic (pairs of) relations introduced in definition 4 of our paper.

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