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Bread and Social Justice: Measurement of Social Welfare and Inequality Using Anthropometrics

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Abstract

We address the question of the measurement of health achievement and inequality in the context of variables exhibiting an inverted-U relation with health and well-being. The chosen approach is to measure separately achievement and inequality in the health increasing range of the variable, from a lower survival bound a to an optimum value m, and in the health decreasing range from m to an upper survival bound b. Because in the health decreasing range, the equally distributed equivalent value associated with a distribution is decreasing in progressive transfers, the paper introduces appropriate relative and absolute achievement and inequality indices to be used for variables exhibiting a negative association with well-being. We then discuss questions pertaining to consistent measurement across health attainments and shortfalls, as well as the ordering of distributions exhibiting an inverted-U relation with well-being. An illustration of the methodology is provided using a group of five Arab countries.

JEL codes: I14, I15, O5

Keywords: anthropometrics, health achievement and inequality, survival thresholds, Arab countries

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Introduction

The improvement of key health indicators has been a major concern of the development debate for many decades, and remains so today, as formulated for instance in the Millennium Development Goals (2000-2015) and Sustainable Development Goals (2015-2030). Beyond improving the average value of key indicators, it has increasingly been recognized that the shape of the distribution is also in need of attention. There are several reasons for turning our attention to inequality in the distribution of a health variable. In the case of calorie intake for instance, low levels of nutrition are associated with stunting in infants and certain severe deficiencies for adults. High levels of energy intake are also problematic, as they increase the risk of cardiovascular disease and type II diabetes (WHO 2011). Additionally, there are the usual normative concerns for preferring less to more inequality in health, in relation to two distributions with the same mean value. In this context, Wagstaff (2002) forcefully argues that average levels of attainment should not be the prime focus of policy, and instead introduces *health achievement* indices, the analogues of equally distributed equivalent incomes of the Atkinson-Kolm-Sen (AKS) approach, as the appropriate metrics of health policy. Thus, greater emphasis and interest by researchers in the last twenty years has placed the measurement of achievement and inequality in health at the centre of the development debate (Wagstaff 2002, Erreygers 2013).

In this paper we are interested in the context of anthropometric health indicators such as body mass, which exhibit an inverted U relation with health status. Other leading examples of anthropometrics include hip, and mid-upper arm, circumferences. One point we emphasize in this paper, following Aristondo and de la Vega (2013) and Kjellsson and Gerdtham (2013) is that survival considerations place lower and upper bounds on the range of values an anthropometric indicator takes. Following Apablaza et al. (2016), we accommodate the non-monotonic relation between a health indicator y and well-being by measuring social welfare when the health variable is reported on the union of two intervals $(a, m]$, and (m, b), where a and b are respectively lower and higher critical bounds beyond which survival is no longer likely, while m is the optimum level of the health indicator. In the context of anthropometrics such as body mass, the lower bound is generally taken to be $a = 10$ kilograms per square meter, b is approximately equal to 60, while m can generally be any value chosen in the interval of 18.5 to 24.9 (WHO, 2017).

The main emphasis of this paper is on the upper tail of the distribution of the health indicator, the interval from m to b, as the measurement of inequality and achievement is generally well understood in the context of variables that exhibit a positive association with well-being. With this observation in mind, we state the four objectives of this paper. Firstly, we enrich the Wagstaff (2002) framework in deriving health achievement indices in the context of variables that are negatively associated with welfare, from both the perspectives of the distributions of health attainments and shortfalls. Because, in the present context, welfare is decreasing in the health indicator, we show in Proposition 1 that the equally distributed equivalent (EDE) value is a Schur convex function; that is, the EDE value is decreasing in Pigou-Dalton transfers (when the EDE is typically increasing in progressive transfers in the context of variables positively associated with health and well-being). We then address the question of the robustness of the achievement index to changes in the survival thresholds, and we are led to derive a new family of achievement indices that are translation invariant.

Lambert and Zheng (2011) provide a family of theorems clarifying the relation between the inequality ordering of a pair of distributions in terms of health attainments and shortfalls. In the formulation of Lambert and Zheng, y denotes health attainment, and $m - y$ denotes health shortfall. Accordingly, the second objective of this paper is to extend Theorems 1 and 2 of these authors, in the context of our study of the upper tail of the distribution of an anthropometric variable, where health attainment is $b - y$ and health shortfall is $y - m$. A related literature has examined various forms of the consistency property in relation to inequality indices of health attainments and shortfalls (Erreygers 2009; Lambert and Zheng 2011; de la Vega and Aristondo 2012; Aristondo and de la Vega 2013; Chakravarty et al. 2016; Bosmans 2016; Yalonetzky 2020). Hence, we furthermore explore the consistency property in relation to the health achievement indices introduced in the paper.

The third purpose of this paper is to reflect on how to compare a pair of distributions in terms of social welfare and inequality, over the entire domain of a health indicator y that exhibits a non-monotonic relation with well-being. One approach developed in Apablaza et al. (2016) is to aggregate shortfalls and excess values from the optimum. The approach we take in this paper however is different, in that we do not view the contribution to social welfare of an undernourished person i and an overweight person j as being comparable. Instead, the approach we pursue in this paper is to construct an order relation on the entire domain of the health indicator y, using as inputs two separate order relations – one of which is defined on the lower tail of the distribution, and the other on the upper tail. We suggest two such construction methods, that we call a *product order* and a *lexicographic order*, and explain the implicit value judgments needed to justify the adoption of each of these order relations.

In a recent study, Alvaredo et al. (2019, p.686), conclude that in terms of income concentration, the Middle East and North Africa (MENA) 'appears to be the most unequal region in the world'. The final purpose of this paper is to illustrate the methodology we develop, in the context of a group of five countries from the MENA region (Comoros, Egypt, Jordan, Morocco and Yemen) where we study variations in social welfare and inequality in relation to an anthropometric variable, namely body mass.

The structure of the paper is as follows. In Section 2 we examine issues arising from the measurement of health achievement and inequality in the context of variables exhibiting a negative association with well-being. In Section 3 we focus our discussion on the derivation of absolute achievement and inequality indices, for such variables. In Section 4 we extend the Lambert and Zheng (2011) theorems, in the context of variables that exhibit a negative association with well-being. Section 5 discusses the ordering of distributions exhibiting an inverted-U relation with well-being. Section 6 illustrates the methodology of the paper using a group of five Arab countries, and Section 7 concludes the paper. Further results and proofs are relegated to two appendices.

2. Relative achievement and inequality measurement with an indicator negatively associated with health

We consider anthropometric measures of health such as the body mass index (BMI), which have two defining properties: (i) survival places lower and upper bounds on their domain of variation, and (ii) they exhibit a non-monotonic, inverted U, relation with health.[1] Let $\Omega = (\omega_1, \dots, \omega_n)$ be a vector of anthropometric observations where ω_i is defined on one of two intervals: either $\omega_i \in (a, m]$ or $\omega_i \in (m, b)$. The distribution $\Omega = (\omega_1, \dots, \omega_n)$ can, thus, be partitioned into two separate vectors defined, respectively, over the intervals (m, b) and $(a, m]$.

Let n_1 and $n_2 = n - n_1$ denote respectively, the number of observations that belong to the two intervals (m, b) and $(a, m]$. We may now define two vectors $Y \coloneqq (y_1, \dots, y_{n_1})$ and $X \coloneqq$

[1] Note that this methodology could be adapted in other settings. Consider the context of routinely used biomarkers of health status, such as sugar level. Then, the lower bound for survival is $a = 40$ milligrams of glucose per decilitre of blood, and the corresponding critical upper bound is $b = 450$ milligrams per deciliter.

$(x_1, \dots, x_{n_2})$ such that $\Omega = [X\ Y]$.[2] Throughout, we work with vectors X and Y that exhibit some variation. Accordingly, we define the following sets

$$D_{mb} = \{Y \in (m,b)^{n_1}: y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n_1)} \; and \; y_{(1)} < y_{(n_1)}\}$$

$$D_{am} = \{X \in (a,m)^{n_2}: x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n_2)} \; and \; x_{(1)} < x_{(n_2)}\}$$

where $Y \uparrow = \left(y_{(1)}, \dots, y_{(n)}\right)$ and $X \uparrow$ are the increasing rearrangements of the vector Y and X; and we restrict our attentions to vectors X and Y defined respectively on D_{am} and D_{mb}.

In the *Atkinson-Kolm-Sen* (AKS) approach, the derivation of inequality indices is approached in relation to a social welfare function taken to capture a preference for higher health, and less inequality. The inequality index is derived via a comparison of the mean of a variable with the equally distributed equivalent (EDE) value of the distribution. Health achievement indices (Wagstaff 2002) are also derived as equally distributed values. One purpose of this section is to show that for an anthropometric indicator exhibiting a negative association with health, the equally distributed equivalent value is decreasing in progressive transfers (when we would expect the opposite from such a summary statistic). We then dwell further on the implications of this finding for alternative specifications of achievement and inequality indices.

2.1 Fundamental axioms

We provide an axiomatic formulation of the measurement of social welfare in relation to health attainment, and we discuss more briefly the form of the underlying social welfare function in relation to health shortfall. With respect to health attainment, we measure welfare with reference to individual i's position from the upper survival threshold b using a social valuation function,

[2] Let $a = 10, m = 18.5,\ b = 60$ and consider the vector $\Omega = (12\ \ 15\ \ 21\ \ 40\ \ 50)$. Then n_1 =3, and $n_2 = 2$ denote, respectively, the number of observations that belong to the two intervals (m, b) and $(a, m]$. We may now define two vectors $Y \coloneqq (21\ \ 40\ \ 50)$ and $X \coloneqq (12\ \ 15)$ so that $\Omega = [X\ Y]$ as required.

more simply a utility function, $\phi_i(b - y)$ for $i = 1, \dots, n_1$, and we let $W^Y: D_{mb} \to \mathbb{R}$ denote a social welfare function in relation to an anthropometric variable y negatively related with health. We let ι_{n_1} denote an n_1-dimensional vector of ones, $\iota_{n_1} = (1, \dots, 1)$, and we consider several axioms commonly used for social welfare functions. In what follows therefore $b\iota_{n_1} - Y$ is a compact notation for the vector $(b - y_1, \dots, b - y_{n_1})$. We begin by stating some elementary transformations of the data.

Definition 1 *Let $Y^A = (y_1^A, \dots, y_{n_1}^A)$ and $Y^B = (y_1^B, \dots, y_{n_1}^B)$ denote two distributions in D_{mb} .*

i. We say that Y^B is obtained from Y^A using a single increment on D_{mb} if for some person i and $\varepsilon \in \left(0, b - y_i^A\right)$, $y_i^B = y_i^A + \varepsilon$ and $y_j^B = y_j^A$ for all $j \neq i$.

ii. We say that Y^B is obtained from Y^A using a single decrement on D_{mb} if for some person i and $\varepsilon \in \left(0, y_i^A - m\right)$, $y_i^B = y_i^A - \varepsilon$ and $y_j^B = y_j^A$ for all $j \neq i$.

iii. We say that Y^B is obtained from Y^A using a single Pigou-Dalton transfer on D_{mb} if there are individuals i and j with $y_i^A < y_j^A$, and $\delta < (y_j^A - y_i^A)/2$, $y_i^B = y_i^A + \delta$, $y_j^B = y_j^A - \delta$ and $y_l^B = y_l^A$ for all $l \neq i, j$.

We next consider the set of social welfare functions on D_{mb} that are anonymous, and defined as the average of welfare levels experienced by individuals:

- ADD (Additivity): The social welfare function is the average of the utility levels of the n_1 individuals:

$$W^Y\left(b - y_1, \dots, b - y_{n_1}\right) \coloneqq \frac{1}{n_1} \sum_{i=1}^{n_1} \phi(b - y_i) \tag{1}$$

where the function $\phi: (m, b) \to \mathbb{R}$ is the anonymous utility function.

The monotonicity axiom MON requires that decrements of Definition 1.ii increase social welfare; that is social welfare increases when individual endowments y_i are reduced. Preference for greater equality is introduced via the axiom $EQUAL$, requiring that social welfare strictly increases with Pigou-Dalton transfers (see Definition 1. iii).

- MON (Monotonicity): The social welfare function W^Y is strictly decreasing in $y_1, \dots, y_{n_1}$.
- $EQUAL$ (Social aversion to inequality): Let $Y^* = \left(y_1^*, \cdots, y_{n_1}^*\right)$ be obtained from $Y = \left(y_1, \cdots, y_{n_1}\right)$ via one or several Pigou-Dalton transfers, then $W^Y\left(b - y_1^*, \cdots, b - y_{n_1}^*\right) > W^Y\left(b - y_1, \cdots, b - y_{n_1}\right)$. That is, W^Y is strictly increasing in Pigou-Dalton transfers.

The monotonicity axiom MON restricts the derivative of the function ϕ to have a negative sign on the interval (m, b). On the other hand, the social welfare function satisfies the social aversion to inequality axiom, $EQUAL$ if ϕ is concave on (m, b). Finally, we discuss two invariance axioms capturing certain transformations of the data that leave the ordering of distributions by the social welfare function unchanged. The first of these, $SCALINV$, guarantees that the ranking of a pair of distributions does not change when units of measurement are modified in a particular manner. The second, $TRANSINV$, is similarly used to capture the notion that the ranking of distributions is invariant to translational shifts in the data.

- $SCALINV$ (Scale invariance): For any scalar $\lambda > 0$, and for any pair of distributions $Y^A, Y^B \in D_{mb}$, there holds $W^Y\left(b\iota_{n_1} - Y^A\right) \geq W^Y\left(b\iota_{n_1} - Y^B\right) \Leftrightarrow W^Y\left(\lambda b\iota_{n_1} - \lambda Y^A\right) \geq W^Y\left(\lambda b\iota_{n_1} - \lambda Y^A\right)$.

- $TRANSINV$ (Translation invariance): For any admissible value of λ and for any pair of distributions $Y^A, Y^B \in D_{mb}$, $W^Y\left(b\iota_{n_1} - Y^A\right) \geq W^Y\left(b\iota_{n_1} - Y^B\right) \Leftrightarrow W^Y\left(b\iota_{n_1} - \left(Y^A + \lambda\iota_{n_1}\right)\right) \geq W^Y\left(b\iota_{n_1} - \left(Y^B + \lambda\iota_{n_1}\right)\right)$.[3,4]

Together the axioms ADD, MON, $EQUAL$ and $SCALINV$ restrict the choice of $\phi(b-y)$ to the family $u_\beta(b-y)$ of power functions (see for instance Kolm, 1976 or Skiadas, 2013):

$$u_\beta(b-y) = \begin{cases} \dfrac{(b-y)^{1-\beta}}{1-\beta}, & \beta > 0, \beta \neq 1 \\ \ln(b-y), & \beta = 1 \end{cases} \tag{2}$$

Accordingly, the family of social welfare functions that satisfies the above five axioms is of the form:

$$W_\beta^Y\left(b-y_1, \dots, b-y_{n_1}\right) = \frac{1}{n_1}\sum_{i=1}^{n_1} u_\beta(b-y_i) \tag{3}$$

We shall return to the axiom $TRANSINV$ in the next section of the paper.

2.2 Anthropometric relative achievement index

Let $\hat{y} \in (m, b)$ denote the equally distributed health level in the distribution Y: it then follows that $u_\beta(b-\hat{y}) = W_\beta^Y\left(b-y_1, \dots, b-y_{n_1}\right)$. Following Wagstaff (2002), this equally distributed equivalent value is known as the *achievement index* associated with the distribution of health attainments. In the income inequality literature, the equally distributed equivalent income is

[3] In this context, a value of λ is admissible if $Y + \lambda\iota_{n_1}$ is an element of D_{mb}, whenever Y is an element of D_{mb}.

[4] Consider the following formulation of the translation invariance axiom: for any admissible value of λ and for any pair of distributions $Y^A, Y^B \in D_{mb}$, $W^Y\left(b\iota_{n_1} - Y^A\right) \geq W^Y\left(b\iota_{n_1} - Y^B\right) \Leftrightarrow W^Y\left((b+\lambda)\iota_{n_1} - (Y^A + \lambda\iota_{n_1})\right) \geq W^Y\left((b+\lambda)\iota_{n_1} - (Y^B + \lambda\iota_{n_1})\right)$. Upon simplifying the right-hand side of the equivalence, we would have that

$$W^Y\left(b\iota_{n_1} - Y^A\right) \geq W^Y\left(b\iota_{n_1} - Y^B\right) \Leftrightarrow W^Y\left(b\iota_{n_1} - Y^A\right) \geq W^Y\left(b\iota_{n_1} - (Y^B)\right)$$

Therefore, this formulation of the axiom would not entail any particular restrictions on the form of the social welfare function.

increasing in Pigou-Dalton progressive transfers. The context of health indicators exhibiting a negative association with welfare produces a marked difference:

Proposition 1 *Let $u()$ denote any real-valued function that is strictly decreasing, and concave on some closed interval $[m^0, b^0] \subseteq (m, b)$. Then, for any distribution $Y \in D_{mb}$, with mean $\bar{y}$, the equally distributed equivalent value*

$$\hat{y} = u^{-1}\left(\frac{1}{n_1}\sum_{i=1}^{n_1} u(y_i)\right) \tag{4}$$

is a Schur-convex function; that is, $\hat{y}$ is decreasing in Pigou-Dalton transfers. Furthermore, $\hat{y}$ satisfies the inequality $m \leq \bar{y} \leq \hat{y}$.

We shall make repeated use of the above result (see Corollaries 2 and 3 below). Because, on the one hand, the equally distributed equivalent value is a Schur-convex function, and on the other hand, the inequality between the mean income and the equally distributed equivalent is reversed, we shall have to write AKS inequality indices based on $\hat{y}$ in a different form in the context of distributions defined on D_{mb}.

Returning to the specific context of Eq. 2 and 3, the equally distributed equivalent value is of the form:

$$\hat{y}_R^a(Y;\beta) = \begin{cases} b - \left(\frac{1}{n_1}\sum_{i=1}^{n_1}(b - y_i)^{1-\beta}\right)^{\frac{1}{1-\beta}}, & \beta > 0, \beta \neq 1 \\ b - \exp\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\ln(b - y_i)\right), & \beta = 1 \end{cases} \tag{5}$$

We refer to $\hat{y}_R^a(Y;\beta)$ as the *anthropometric relative achievement index of the distribution of health attainments*. The superscript a in $\hat{y}_R^a$ is introduced to denote that the equally distributed equivalent value is associated with a concept of health attainment. The subscript R in $\hat{y}_R^a$ is introduced to denote relative achievement indices; i.e. indices that satisfy the scale invariance axiom $SCALINV$. Note that because an increase in y is associated with a decrease in health and well-being, *larger* values of $\hat{y}_R^a$ are associated with *lower* levels of social welfare.

As an alternative to measuring social welfare in relation to health attainment, we briefly discuss the context of shortfall $y - m$. We associate a social welfare function $W^Y\big(y_1 - m, \dots, y_{n_1} - m\big)$ with the vector of shortfalls $Y - m\iota_{n_1}$. The axioms of additivity, monotonicity, preference for equality and scale invariance are easily defined in this context. They imply that the social welfare function is the average of utilities, where the individual utility function is of the form:

$$u_\gamma(y - m) = -\frac{(y - m)^{1-\gamma}}{1 - \gamma}; \qquad \gamma < 0 \tag{6}$$

The function $u_\gamma()$ is decreasing in y and concave. The associated equally distributed equivalent value in terms of health shortfalls is of the present form

$$\hat{y}_R^s(Y;\gamma) = m + \sum_{i=1}^{n_1}\left(\frac{1}{n_1}\sum_i (y_i - m)^{1-\gamma}\right)^{\frac{1}{1-\gamma}}; \quad \gamma < 0. \tag{7}$$

As an application of Proposition 1 we state the following result:

Corollary 2: *For $Y \in D_{mb}$, each of the equally distributed equivalent values $\hat{y}_R^a$ and $\hat{y}_R^s$ associated respectively with the distribution of health attainments and shortfalls is a Schur-convex function. Furthermore, the following inequalities are satisfied: $m \leq \bar{y} \leq \hat{y}_R^a$ and $m \leq \bar{y} \leq \hat{y}_R^s$.*

2.3 The AKS family of relative inequality indices

It follows from Proposition 1 and Corollary 2 that application of the standard AKS inequality index introduced by Atkinson (1970), namely the function $1 - (\hat{y}_R^a/\bar{y})$ in the context of attainment, and $1 - (\hat{y}_R^s/\bar{y})$ in the context of shortfalls, will provide the data user with two challenges. Firstly, in the light of the inequalities stated in Proposition 1 each of these two inequality indices will take on negative values. More importantly, Pigou-Dalton transfers will increase the value taken by these indices. It is thus important to adapt the relative AKS index in the context of anthropometric indicators exhibiting a negative association with health, so as to achieve these desired properties (positive-valued function, and decreasing in Pigou-Dalton transfers). Consider in particular the following forms:

$$I_R^a(Y;\beta) \coloneqq 1 - \left(\frac{b - \hat{y}_R^a}{b - \bar{y}}\right) \quad (8.a)$$

$$I_R^s(Y;\gamma) \coloneqq 1 - \left(\frac{\bar{y} - m}{\hat{y}_R^s - m}\right) \quad (8.b)$$

Because of the inequalities $m \leq \bar{y} \leq \hat{y}_R^a$ and $m \leq \bar{y} \leq \hat{y}_R^s$, each of the indices (8.a) and (8.b) takes on positive values. Furthermore, because both equally distributed equivalent values $\hat{y}_R^a$ and $\hat{y}_R^s$ are decreasing in Pigou-Dalton transfers, the inequality indices (8.a) and (8.b) are now increasing functions of $\hat{y}_R^a$ and $\hat{y}_R^s$. Finally, it is clear that the inequality indices above are invariant to rescaling b, m and the distribution Y by the same constant $\lambda > 0$.

3. Attainments, shortfalls and the absolute anthropometric achievement index

The scale invariance axiom guarantees that changing the units of measurement of y, and the two thresholds m and b does not result in any change in social welfare and inequality (Eq. 5 and 8). A separate concern however may have to do with disagreement about the level of the thresholds m and b. Medical research can inform about the values of m and b. But in most cases, such information comes in the form of a range of values for the parameters.

Recall that the threshold m enters the calculation of health shortfalls, while the upper survival threshold plays a similar role in the calculation of health attainments. Consider therefore, the effect of changing the value assigned to the upper threshold b, from b^o to $b^1 = b^o - \lambda$. Then, observing that $b^1\iota_{n_1} - Y = (b^o - \lambda\iota_{n_1}) - Y = b^o\iota_{n_1} - (Y + \lambda\iota_{n_1})$, changing the value assigned to the upper threshold is equivalent to obtaining a new distribution $Y + \lambda\iota_{n_1}$, while maintaining the upper threshold at the initial value b^o. It is clear that such translational shifts in the distribution of resources will result in a shift in the relative Lorenz curve. Following Kolm (1976) and Moyes (1987), it is however possible to work with inequality indices that are invariant to changes in the thresholds m and b.

The key to deriving indices that are robust to changes in the upper threshold b is to replace the scale invariance axioms by translation invariance axiom$s$. Consider first the measurement of social welfare in relation to health attainment. Following Kolm (1976), or Skiadas (2013), together the axioms ADD, MON, $EQUAL$ and $TRANSINV$ restrict the choice of $\phi(b-y)$ to the family $u_\kappa(b-y)$ of exponential functions:

$$u_\kappa(b-y) = 1 - \exp\big(-\kappa(b-y)\big), \qquad \kappa > 0 \tag{9}$$

Accordingly, the family of social welfare functions defined on D_{mb} that satisfies the above four axioms is of the form

$$W_\kappa^Y\big(b-y_1, \dots, b-y_{n_1}; m\big) = \frac{1}{n_1}\sum_{i=1}^{n_1} u_\kappa(b-y_i) \tag{10}$$

The equally distributed equivalent value $\hat{y}_A^a$ pertaining to the above family of social welfare functions satisfies the identity $u_\kappa(b-\hat{y}_A^a) = \frac{1}{n_1}\sum_{i=1}^{n_1} u_\kappa(b-y_i)$. Specifically,

$$\hat{y}_A^a(Y;\kappa) = b + \frac{1}{\kappa}\ln\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\exp\big(-\kappa(b-y_i)\big)\right) \tag{11}$$

We refer to $\hat{y}_A^a$ as *the anthropometric absolute achievement index of the distribution of health attainments*. As such, the index (11) complements the work of Wagstaff (2002) that pertained to the derivation of the relative achievement indices in relation to health attainment.

Similarly, switching to shortfalls, $y - m$, the utility function $u_\delta: D_{mb} \longrightarrow \mathbb{R}$ that satisfies the axioms discussed above is of the form

$$u_\delta(y - m) = 1 - \exp\big(-\delta(y - m)\big), \qquad \delta < 0 \tag{12}$$

The resulting *anthropometric absolute achievement index of the distribution of health shortfalls* is given by the expression

$$\hat{y}_A^s(Y;\delta) = m - \frac{1}{\delta}\ln\left[\frac{1}{n_1}\sum_{i=1}^{n_1}\exp\big(-\delta(y_i - m)\big)\right]. \tag{13}$$

As a result of imposing the underlying translation invariance axiom, it is to be noted that the absolute achievement index for health attainment $\hat{y}_A^a$ is invariant to changes in the value assigned to the upper survival threshold b, and likewise, $\hat{y}_A^s$ is invariant to changes in the value taken by the parameter m. Furthermore, the following result is a direct application of Proposition 1.

Corollary 3 *For a distribution $Y \in D_{mb}$, each of the equally distributed equivalent values $\hat{y}_A^a$ and $\hat{y}_A^s$ associated respectively with the distributions of health attainments and shortfalls is a Schur-convex function. Furthermore, the following inequalities are satisfied: $m \leq \bar{y} \leq \hat{y}_A^a$ and $m \leq \bar{y} \leq \hat{y}_A^s$.*

Because the equally distributed equivalent values $\hat{y}_A^a$ and $\hat{y}_A^s$ are decreasing in Pigou-Dalton transfers, we shall presently write the Kolm absolute inequality indices for health attainments and shortfalls in the following forms:

$$I_A^a(Y;\, \kappa) = \hat{y}_A^a - \bar{y} \tag{14.a}$$

$$I_A^s(Y;\, \delta) = \hat{y}_A^s - \bar{y} \tag{14.b}$$

Written in the form (14.a-14.b), the above inequality indices take on positive values, and are decreasing in Pigou-Dalton transfers.

4. From health attainment to shortfalls: the Lambert and Zheng consistency theorems

One important question that Lambert and Zheng (2011) address is to what extent the inequality ordering of two health distributions is robust to the specifications of outcomes in terms of attainments and shortfalls.

The inequality indices (8 and 14) associated with the social welfare functions underlying the derivation of the relative and absolute forms of the anthropometric achievement index do not satisfy the consistency property across the distribution of attainments and shortfalls. That is, as demonstrated by Lambert and Zheng (2011), in the class of rank independent inequality indices that are additively decomposable, only the variance satisfies the consistency property.

As forcefully demonstrated by de la Vega and Aristondo (2012), Bosmans (2016), Chakravarty et al. (2016) and Yalonetzky (2020), it is certainly possible to dispense with some of the axioms of this paper in order to obtain indices of inequality that are consistent across health attainments and shortfalls.[5] As the emphasis in this paper is on social welfare and achievement in relation to anthropometric variables, we first explore how the Lambert and Zheng (2011) theorems, as originally formulated in the context of variables y in positive association with health, need to be extended in the context of variables defined on D_{mb}.

For this purpose, it is convenient to define the Zoli (1999) two-parameter family of Lorenz orderings $\preccurlyeq_{\tau,\theta}$ on D_{mb}, with $0 \leq \tau, \theta \leq 1$. For a given distribution $Y \coloneqq (y_1, \dots, y_{n_1}) \in D_{mb}$, define:

$$d(Y;\theta,\tau) = [\,\theta\mu_Y + (1-\theta)]^{\tau} \tag{15}$$

and

[5] Typically, the definition of consistency one adopts can be relaxed (cf. Propositions 5 and 6). It is also possible to address the issue of consistency in relation to relative shortfalls and attainments. See Bosmans (2016) and Yalonetzky (2020).

$$\ell(y_i; Y, \theta, \tau) = \frac{1}{d(Y; \theta, \tau)}(y_i - \mu_Y) \quad (16)$$

where, for ease of exposition, from here on μ_Y will denote the mean of the vector Y. For given parameter values τ and θ, and for two distributions, Y^A and Y^B in D_{mb}, we will say that Y^A Zoli-dominates Y^B, written $Y^A \succcurlyeq_{\tau,\theta} Y^B$ if the following n_1 inequalities are satisfied:

$$\frac{1}{n_1}\sum_{i=1}^{j} l\left(y_{(i)}^A; Y^A; \theta, \tau\right) \geq \frac{1}{n_1}\sum_{i=1}^{j} l\left(y_{(i)}^B; Y^B; \theta, \tau\right) \;\; \forall j = 1, \dots, n_1 \quad (17)$$

Of particular relevance in this paper are the case $\theta = \tau = 0$, which produces the ordering by absolute Lorenz curves (Moyes 1987), and the case $\theta = \tau = 1$, which produces the relative Lorenz ordering.[6] Let μ_Z and μ_S respectively denote the mean health attainment and shortfall associated with a vector $Y \in D_{mb}$. In the context of health attainments and shortfalls, we also define the following functions of the vectors of shortfalls and attainments

$$\hat{Z}(Y;\, \theta, \tau) = \frac{1}{d(Z; \theta, \tau)}\left(z_1 - \mu_Z, \dots, z_{n_1} - \mu_Z\right) \quad (18)$$

$$\hat{S}(Y; \theta, \tau) = \frac{1}{d(S; \theta, \tau)}\left(s_1 - \mu_S, \dots, s_{n_1} - \mu_S\right) \quad (19)$$

The following results provide an adaptation of Lambert and Zheng's (2011) Theorems 1 and 2 in the context of this paper (see Appendix B for proofs).

Proposition 4 *In the context of anthropometric variables y_i that exhibit a negative association with health, and are subject to survival bounds, define attainment as $z_i = b - y_i$ and shortfall as $s_i = y_i - m$. Consider two distributions, Y^A and Y^B in D_{mb} with respective means μ_{Y^A} and μ_{Y^B} and with associated functions $\hat{Z}(Y^A; \theta, \tau)$ and $\hat{Z}(Y^B; \theta, \tau)$ of the vectors of attainments, and*

[6] For the case where $0 < \tau < 1$ and $\theta = 1$, we obtain in Eq. 17 intermediate inequality orderings (see Lambert and Zheng 2011 for further discussion).

associated functions $\hat{S}(Y^A;\theta,\tau)$ *and* $\hat{S}(Y^B;\theta,\tau)$ *of the vectors of shortfalls. For* $i = A, B$ *let* $\mu_{Z^i} = b - \mu_{Y^i}$. *Then, for* $0 \leq \theta \leq 1$ *and* $0 \leq \tau \leq 1$, *and* $\hat{Z}(Y^A;\theta,\tau) \sim_{\tau,\theta} \hat{Z}(Y^B;\theta,\tau)$,

(a) in the context of the absolute Lorenz order (that is where $\tau\theta = 0$), $\hat{S}(Y^A;\theta,\tau) \sim_{\tau,\theta} \hat{S}(Y^B;\theta,\tau)$.

(b) If $\tau\theta \neq 0$, *then* $\hat{S}(Y^A;\theta,\tau) \succcurlyeq_{\tau,\theta} \hat{S}(Y^B;\theta,\tau)$ *if* $\mu_{Z^A} < \mu_{Z^B}$ *and* $\hat{S}(Y^A;\theta,\tau) \preccurlyeq_{\tau,\theta} \hat{S}(Y^B;\theta,\tau)$ *if* $\mu_{Z^A} > \mu_{Z^B}$.

As discussed in the literature, ranking distributions in terms of health shortfalls versus attainments does not in general produce consistent comparisons within the Zoli parametric family of orderings. Only in the case of the absolute Lorenz order, statement (a) above reveals, that as in Lambert and Zheng (2011), in the present formulation, two distributions Y^A and Y^B are ranked equivalent in terms of health attainment if and only if they are equivalent in terms of health shortfalls. On the other hand, as point (b) reveals, outside of the context of the absolute Lorenz order, if the two distributions are equivalent in terms of attainments, the relative magnitude of the average attainments in the two distributions determines the ranking of the distributions of shortfalls.

Proposition 5 *The absolute Lorenz ordering is consistent across health attainments and shortfalls. That is, for the absolute inequality ordering* ($\theta = 0$ *and* $0 \leq \tau \leq 1$), *and for any* Y^A *and* Y^B *in* D_{mb} *we have* $\hat{S}(Y^A;0,\tau) \succcurlyeq_{\tau,\theta} \hat{S}(Y^B;\ 0,\tau) \Leftrightarrow \hat{Z}(Y^A;0,\tau) \succcurlyeq_{\tau,\theta} \hat{Z}(Y^B;0,\tau)$.

Proposition 5 has the following implication for the empirical section of the paper: if say the distribution of health attainments in Morocco is more egalitarian (in the sense of the absolute Lorenz order) than the distribution of, say Jordan, then it is also the case that in the context of health shortfalls, the Morocco distribution is more egalitarian than the Jordan distribution.

Because we shall make extensive use of the generalized Lorenz curve in the empirical section of the paper, we have found it useful to state an appropriate consistency result across health attainment and shortfalls in the context of this order relation. For $i = A, B$ define the attainment and shortfall vectors $Z^i = b\iota_{n_1} - Y^i$ and $S^i = Y^i - m\iota_{n_1}$. For the increasing rearrangement $Z\uparrow = (z_{(1)}, \dots, z_{(n_1)})$, following Shorrocks (1983), Z^B is generalized Lorenz dominated by Z^A, written as $Z^B \prec_w Z^A$ if the following n_1 inequalities are satisfied:

$$\sum_{i=1}^{j} z_{(i)}^B \leq \sum_{i=1}^{j} z_{(i)}^A \; ; \qquad j = 1,2, \dots, n_1 \tag{20}$$

Then, the distribution of attainments Z^A generalize-Lorenz dominates Z^B if Z^A is obtained from Z^B via a sequence of increments[7] and Pigou-Dalton transfers. Accordingly, we further introduce a relation $\prec_{SM}$ on the distribution of shortfalls. For Y^A, $Y^B \in D_{mb}$ we shall write $S^A \prec_{SM} S^B$, if S^B is obtainable from S^A using a sequence of increments and regressive transfers. We view the relation $\prec_{SM}$, defined on shortfall distributions, as the dual to the relation $\prec_w$ defined on attainments distributions. We refer to $\prec_{SM}$ as *shortfall majorization.*[8] Define $Y\downarrow = (y_{[1]}, y_{[2]}, \dots y_{[n_1]})$ as the decreasing arrangement of the vector Y. Following Marshall et al. (2011, chapter 1), $S^A \prec_{SM} S^B$ if the following n_1 inequalities are satisfied:

$$\sum_{i=1}^{j} (y_{[i]}^A - m) \leq \sum_{i=1}^{j} (y_{[i]}^B - m) \; ; \qquad j = 1,2, \dots, n_1 \tag{21}$$

We are now ready to state the following result:

Proposition 6 *Let $Y^A, Y^B \in D_{mb}$. Then, there holds $S^A \prec_{SM} S^B$ if and only if $Z^B \prec_w Z^A$. Equivalently, the distribution of health shortfalls S^A is shortfall-majorized by the distribution of health shortfalls S^B, if and only if the distribution of health attainments Z^A generalize- Lorenz dominates the distribution of health attainments Z^B.*

[7] Note that in D_{mb}, attainment increases when y decreases.
[8] We note nonetheless that in other disciplines $\prec_{SM}$ is known under the name of *weak sub-majorization* (see Marshall et al. 2011, chapter 1).

Observe that unlike the Lambert and Zheng (2011) consistency theorems that are formulated in relation to a fixed ordering across shortfalls and attainments, we have used two separate orderings, namely, $\prec_{SM}$ in relation to shortfalls, and $\prec_{w}$ in relation to health attainments, in order to state the above consistency result. As such, we have followed a more flexible approach as advocated by Bosmans (2016) in deriving a practically relevant consistency result for the distributional analysis of health attainment and shortfalls in relation to social welfare.[9]

We next state a further consistency result, in relation to the anthropometric absolute achievement index. For ease of exposition, instead of Y^A, Y^B, we now write Y^1, Y^2 for the pair of distributions to be compared.

Proposition 7 *Let* $Y^1, Y^2 \in D_{mb}$ *denote two distributions. Associate with* Y^j*, an anthropometric absolute achievement index* $\hat{y}_A^a(Y^j;\, \kappa)$ *for the distribution of health attainments and an index* $\hat{y}_A^s(Y^j; \delta)$ *for the distribution of health shortfalls. Then, for any value* $\kappa > 0$ *we have that* $\hat{y}_A^a(Y^1; \kappa) \geq \hat{y}_A^a(Y^2; \kappa)$ *if and only if* $\hat{y}_A^s(Y^1; -\kappa) \geq \hat{y}_A^s(Y^2; -\kappa)$.

Observe that unlike Theorem 4 of Lambert and Zheng (2011), we obtain the above consistency result in relation to a pair of distributions *and* a pair of absolute anthropometric indices. Note furthermore, that while we have stated Proposition 7 in terms of achievement indices, from Eqs. 14.a and 14.b it is possible to obtain a comparable result in association with the absolute inequality indices for health attainments and shortfalls: that is, for any pair of distributions $Y^1, Y^2 \in D_{mb}$ and any value $\kappa > 0$ we have that $I_A^a(Y^1; \kappa) \geq I_A^a(Y^2; \kappa)$ if and only if $I_A^s(Y^1; -\kappa) \geq I_A^s(Y^2; -\kappa)$.

[9] Note furthermore that because the n_1 inequalities defining the generalized Lorenz ordering are linear in the distributions Y^A and Y^B, when working with health attainments, the ordering of distributions is robust to changes in the values taken by the upper bound b. For the same reason, when working with shortfalls, the generalized Lorenz ordering of a pair of distributions is likewise robust to changes in the values taken by the optimum health threshold m.

To conclude this section, we summarize the implications of the above consistency results for empirical work. Firstly, in terms of inequality orderings, to achieve consistency in distributional comparisons across attainments and shortfalls, we must work with the absolute Lorenz ordering (Proposition 5). To achieve consistency in terms of welfare orderings, the generalized Lorenz ordering in association with health attainments, is consistent with the ordering of distributions by the relation $\prec_{SM}$ in association with health shortfalls. Finally, every absolute achievement index $\hat{y}_A^a(Y;\, \kappa)$ defined on attainments is paired with an achievement index $\hat{y}_A^s(Y; -\kappa)$ defined on shortfalls, rendering the ordering of distributions of health attainments and shortfalls consistent, in the sense of Proposition 7.

5. From orders on the lower and upper tails to full distributional comparisons

We have thus far discussed the question of distributional comparisons in D_{mb}. In Appendix A, we briefly discuss the derivation of relative and absolute achievement indices pertaining to the lower tail of the distribution, D_{am}. As argued earlier, we do not assume that well-being is meaningfully comparable below and above the optimum value, m, in the context of anthropometrics. Consider then two individuals i and j with $x_i < m$ and $y_j > m$ such that $x_i - a = b - y_j$. In contrast with Apablaza et al. (2016), we take the view that despite the fact that $x_i - a = b - y_j$, the welfare levels of two individuals i and j, one being undernourished, and the other overweight are not meaningfully comparable. Therefore, we suggest to construct the social welfare and inequality relations on the entire domain, from two separate order relations defined on respectively the lower and upper tails $(a, m]$ and (m, b) of the distribution of the anthropometric variable. We detail two such procedures. We first begin with the following definitions (see for instance Davey and Priestley, 2010).

Definition 2 *Let* $(P, \preccurlyeq_P)$ *and* $(Q, \preccurlyeq_Q)$ *denote two pre-ordered sets. For all* $(X^A, Y^A), (X^B, Y^B) \in P \times Q$,

(i) *we define the lexicographic order relation* $(P \times Q, \preccurlyeq_{LEX})$ *by* $(X^A, Y^A) \preccurlyeq_{LEX} (X^B, Y^B)$ *if* $X^A \prec_P X^B$, *or if* $X^A \sim_P X^B$ *and* $Y^A \preccurlyeq_Q Y^B$,

(ii) *we define the product order relation* $(P \times Q, \preccurlyeq_\pi)$ *by* $(X^A, Y^A) \preccurlyeq_\pi (X^B, Y^B)$ *if both* $X^A \preccurlyeq_P X^B$ *and* $Y^A \preccurlyeq_Q Y^B$.

Under (i), the lexicographic order, there are two ways to achieve an ordering of two distributions. Firstly, distribution (X^A, Y^A) has lower social welfare than (X^B, Y^B) if it is the case that in the relation $\prec_P$, X^A is dominated by X^B, (regardless of how Y^A and Y^B relate in the relation $\prec_Q$). Secondly, the two distributions may be lexicographically ordered if it is the case that in the $\prec_P$ relation, the social planner is indifferent between X^A and X^B, and that in the relation $\prec_Q$, Y^B is socially preferred over Y^A. Under (ii), the product order, distribution (X^B, Y^B) has higher social welfare than (X^A, Y^A) if it is the case that both X^B is preferred to X^A according to the relation $\prec_P$ and similarly that $Y^A \preccurlyeq_Q Y^B$.

The above lexicographic order assumes implicitly that one domain of the distribution has priority over the other. If reducing undernutrition is a more important social objective than tackling overweight, the set P in part (i) of the definition would denote the lower tail of the distribution, while Q would denote the upper tail of the distribution. If tackling overweight is a social priority over undernutrition, this would amount to swapping around the definitions of the two sets P and Q. We consider both perspectives in our empirical illustrations below. It is finally possible that society is indecisive about the order of priority of tackling inequality in undernutrition and overweight. In this case the product order is an appropriate aggregation procedure for the separate orders, in the sense that it does not consider one tail of the distribution to be more important than the other in terms of society's preferences.

6. An empirical illustration

The purpose of this section is to assess health achievement and inequality using anthropometric data on body mass, pertaining to five Arab countries. In Section 6.1 we compare the five countries in terms of the anthropometric achievement index for health attainments and shortfalls, in its relative and absolute variants, as developed in Sections 2 and 3. Next in Section 6.2, we explore social welfare and inequality orderings pertaining to the upper tail of the distribution of body mass. In Section 6.3 we illustrate the two orderings that we discussed in Section 5, namely the product and lexicographic orders.

Our application makes use of anthropometric data on adult (non-pregnant) women of reproductive age (15 to 49). The analysis is performed using data from Demographic and Health Surveys (DHS) conducted in five Arab countries: Egypt (2015), Yemen (2013), Jordan (2012), Comoros (2012) and Morocco (2004). The anthropometric indicator of interest here is taken as the body mass index (BMI), calculated by the authors as weight in kilograms divided by squared height measured in meters. In line with the guidelines of the World Health Organization (2017), for the purpose of the present analysis, we set the value of a to be equal to 10 and b to be equal to 60, while the cut-off value m is fixed at 24.90.[10] After cleaning the data for missing and miscoded values on the variable of interest, the respective sample sizes are as follows: Egypt ($n_1 = 5226, n_2 = 1962$), Jordan ($n_1 = 6336, n_2 = 4740$), Morocco ($n_1 = 6238, n_2 = 10677$), Yemen ($n_1 = 5666, n_2 = 17276$) and Comoros ($n_1 = 1926, n_2 = 3156$).

[10] For most anthropometrics, m could be defined as a range of values. The WHO guidelines for non-pregnant women define an optimum interval with values of m ranging between 18.5 and 24.9. As this does not raise unresolved conceptual problems in the context of our paper, we shall simplify our exposition by assuming that m is a single point.

6.1 Anthropometric achievement and inequality indices: the upper tail of the distribution

Consider first the measurement of achievement and inequality in relation to health attainments and shortfalls in the upper tail of the distribution. We begin by examining the relative achievement indices $\hat{y}_R^a$ and the related inequality indices I_R^a in the five countries.

We report in Table 1.A calculations pertaining to inequality and achievement indices in relation to three values for the inequality aversion parameter: $\beta = 0.5,\ 1$ and 3. To begin with, it is worth noting that the mean of the distribution is highest in Egypt (32.4) and lowest in Morocco (28.8). Consider first the results pertaining to $\beta = 1$. Recalling that achievement (welfare) is decreasing in y, we find that the anthropometric achievement index $\hat{y}_R^a$ (Eq. 5) ranks social welfare as lowest in the Egypt distribution ($\hat{y}_R^a = 33.2$) followed by Jordan ($\hat{y}_R^a = 31.6$), Comoros ($\hat{y}_R^a = 29.7$), Yemen ($\hat{y}_R^a = 29.5$), while it is highest in Morocco ($\hat{y}_R^a = 29.1$). Increasing the social aversion to inequality parameter ($\beta = 3$), results in lower health achievement (that is, higher values) in all countries. We note nonetheless that this does not change the ranking of the countries.

[Insert Table 1.A Here]

Turning now to inequality of attainments (Eq. 8.a), we find that the relative inequality index, I_R^a , evaluated at $\beta = 1$, takes the highest value in Egypt: 2.9%. In contrast, this figure is the lowest in Morocco (0.8%), while inequality is between these two values in the context of the other three countries. For $\beta = 3$, inequality in Egypt remains highest, at 30.7%. This is followed by 8.9% in Jordan, 5.3% in the Comoros and 4.1% in Yemen, while it is the lowest in Morocco (2.9%).

Turning to the distribution of shortfalls, we report in Table 1.B results pertaining to the achievement and inequality indices $\hat{y}_R^s$ (Eq. 7) and I_R^s (Eq. 8.b). The results of Table 1.B show that the achievement rankings of the countries in terms of the distribution of health shortfall are similar to the results of Table 1.A, pertaining to the distribution of health attainments. However, when we examine inequality indices, there are substantial variations in the rankings of countries when moving from the distribution of health attainments to the distribution of health shortfalls. For example, consider in Table 1.B the results pertaining to $\gamma = -1$. We find that Egypt exhibits the lowest level of inequality in the distribution of shortfalls. On the other hand, the results of Table 1.A indicate that Egypt exhibits the highest level of inequality in the distribution of health attainments.

[Insert Table 1.B here]

Recall that the relative indices of inequality emerged to be inconsistent in their rankings of countries in the distributions of health attainments and shortfalls. We next compare the group of five countries using the absolute achievement and inequality indices; namely $\hat{y}_A^a$ (Eq. 11) and I_A^a (Eq. 14.a). We report in Table 2 findings pertaining to the distribution of health attainments. We discuss briefly the findings related to the inequality aversion parameter $\kappa = 1$. Health achievement remains the lowest in Egypt ($\hat{y}_A^a = 51.7$) and the highest in Morocco ($\hat{y}_A^a = 42.9$). Similarly, inequality remains highest in Egypt ($I_A^a = 19.3$) and lowest in Morocco ($I_A^a = 14.0$). Recall that absolute indices of achievement and inequality are consistent – in the sense of Proposition 7 – in their orderings of distributions of attainments and shortfalls. The results pertaining to the absolute achievement and inequality indices for the distribution of health shortfalls, $\hat{y}_A^s$ (Eq. 13) and I_A^s (Eq. 14.b) confirm this consistency result, and accordingly are not reported (for brevity).

[Insert Table 2 here]

We next report the ordering of countries according to the absolute inequality criteria. In accordance with the Lambert and Zheng (2011) theorems as well as Propositions 4 and 5, we limit ourselves to examining the distribution of health attainments. We additionally recall that the absolute Lorenz order is robust to the choice of the upper survival threshold b. With these observations in mind, we find that the five countries exhibit no intersection between curves, with Morocco exhibiting the most egalitarian distribution, and Egypt the least egalitarian distribution. Between Morocco and Egypt, we find that Yemen comes second followed by Comoros, followed by Jordan.

[Insert Figure 1 here]

6.2 Social welfare orderings

Recall that the generalized Lorenz curve is consistent across rankings of distributions of health attainments and shortfalls in the sense of Proposition 6. In what follows therefore we focus entirely on the distribution of health attainments. In the upper tail of the distribution, the generalized Lorenz curve of a hypothetical optimum health distribution $Y^* = (m, \dots, m)$, would take the form of a straight line starting at zero with a slope equal to $b - m$. We observe on the basis of Figure 2 that the attainment distributions pertaining to the five countries are ordered, in that we do not observe crossing curves. In terms of social welfare, Morocco ranks best, followed by Yemen, then by Comoros. The Jordan distribution is second from the bottom while the Egypt distribution exhibits the least level of social welfare.

We next turn to an examination of the lower tail of the distribution, where health and well-being are positively associated. Figure 3 reveals that Egypt ranks highest in terms of social welfare and Yemen exhibits the least level of social welfare. While the generalized Lorenz curves pertaining to Morocco, Jordan and Comoros appear to overlap, the Morocco distribution generalize-Lorenz dominates the other two distributions. For Jordan and Comoros, we find however that the two curves intersect once, with the Jordan curve lying above the Comoros curve in the first four deciles of the distribution.

[Insert Figures 2 and 3 here]

6.3 Health achievement and social welfare: orderings on the entire domain of the distribution

We focus here our discussion on social welfare, where we first consider the lexicographic order. Firstly, if we are of the view that tackling undernutrition should have precedence over tackling overweight, then the ordering of the five countries is as depicted in the Hasse diagram on the left-hand side of Figure 4. Conversely, if we take the view that tackling overweight should have precedence over tackling undernutrition, the ordering of the distributions is as depicted in the Hasse diagram on the right-hand side of Figure 4. This figure reveals that the approach one takes in the lexicographic order matters for the ranking of countries: Egypt ranks highest and Yemen ranks lowest when the lower tail of the distribution has precedence over the upper tail. Conversely, Morocco ranks best and Egypt ranks lowest when the upper tail has precedence over the lower tail of the distribution.

In Figure 5, we report the Hasse diagram of the five countries from the perspective of the product order. As the product order treats the two domains of the distribution symmetrically, we expect, that overall, it enables us to compare fewer pairs of distributions than either of the lexicographic orders. Indeed, we find that Egypt is not comparable to the other four countries. Furthermore, Yemen, Jordan and Comoros are pairwise incomparable. Nonetheless, these three countries are dominated by the Morocco distribution, from the perspective of the product order.

7. Conclusion

The purpose of the paper was to address the question of the measurement of health achievement and inequality in the context of variables exhibiting an inverted-U relation with well-being. We adopted a general framework whereby we measure separately achievement and inequality in the health-increasing range of the variable, from a lower survival bound a to an optimum value m, and in the health decreasing range from m to an upper survival bound b.

We have shown in Proposition 1 that for variables exhibiting a negative association with well-being, the equally distributed equivalent value is a Schur-convex function: that is, a function that is decreasing in progressive transfers. This has meant that the Wagstaff (2002) health achievement index, and relative Atkinson-Kolm-Sen inequality indices available from the income inequality literature, as well as the Kolm type absolute indices required some adaptation in the context of variables exhibiting a decreasing relation with well-being.

In the empirical illustration, it was found that in the lower tail of the distribution health achievement is highest in Egypt and lowest in Yemen; while in the upper tail of the distribution achievement is highest in Morocco, but lowest in Egypt. That is, while Egypt was found to achieve the highest level of social welfare in relation to the phenomenon of undernutrition, it ranked worst in terms of the distribution of excess body mass. This result points to the need of disaggregating the distributional analysis of an anthropometric health indicator that exhibits a non-monotonic relation with well-being, as we have suggested in this paper.

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References

Apablaza, M., Bresson, F., and Yalonetzky, G. (2016). When more does not necessarily mean better: Health-related illfare comparisons with nonmonotone wellbeing relationship. Review of Income and Wealth, 62(S1), S145-S178.

Aristondo O, and Lasso de la Vega C. (2013). Measuring the inequality of bounded distributions: a joint analysis of attainments and shortfalls, In Health and Inequality, Research on Economic Inequality, Rosa Dias P, O'Donnell O (eds.), vol. 21. Emerald Group Publishing Limited; 33–52.

Atkinson, A. B. (1970). On the measurement of inequality. Journal of Economic Theory, 2(1970), 244-263.

Alvaredo, F., Assouad, L., and Piketty, T. (2019). Measuring lnequality in the Middle East 1990–2016: The World's Most Unequal Region? Review of Income and Wealth, 65(4), 685-711.

Bosmans, K. (2016). Consistent comparisons of attainment and shortfall inequality: a critical examination. Health Economics, 25(11), 1425-1432.

Chakravarty, S. R., Chattopadhyay, N., and D'Ambrosio, C. (2016). On a family of achievement and shortfall inequality indices. Health Economics, 25(12), 1503-1513.

Davey, B. A., and Priestley, H. A. (2002). Introduction to lattices and order. Cambridge University Press.

de la Vega, C. L., and Aristondo, O. (2012). Proposing indicators to measure achievement and shortfall inequality consistently. Journal of Health Economics, 31(4), 578-583.

Erreygers, G. (2009). Can a single indicator measure both attainment and shortfall inequality? Journal of Health Economics, 28(4), 885-893.

Erreygers, G. (2013). A dual Atkinson measure of socioeconomic inequality of health. Health Economics 22(4), 466-479.

Kjellsson, G., and Gerdtham, U. G. (2013). On correcting the concentration index for binary variables. Journal of Health Economics, 32(3), 659-670.

Kolm S-C. (1976). Unequal Inequalities I. Journal of Economic Theory, 12(1976), 416-442.

Lambert, P., and Zheng, B. (2011). On the consistent measurement of attainment and shortfall inequality. Journal of Health Economics, 30(1), 214-219.

Marshall A. W., Olkin I., and Arnold, B. C. (2011). Inequalities: the theory of majorization and its applications (2nd ed.). Springer International Publishing AG., Springer Series in Statistics.

Moyes, P. (1987). A new concept of Lorenz domination. Economics Letters 23 (2) 203-207.

Skiadas C. (2013). Scale-invariant uncertainty-averse preferences and source-dependent constant relative risk aversion, Theoretical Economics, 8, 59-93.

Wagstaff, A. (2002). Inequality aversion, health inequalities and health achievement. Journal of Health Economics, 21: 627-641.

World Health Organization (2011). Regional strategy on nutrition 2010-2019. World Health Organization. Eastern Mediterranean Regional office (EMRO), Cairo, Egypt.

World Health Organization (2017). Global Database for Body Mass Index. http://apps.who.int/bmi/index.jsp?introPage=intro 3.html.

Yalonetzky, G. (2020). Inequality of ratios. METRON 78.2:193-217.

Zoli, C. (1999). A generalized version of the inequality equivalence criterion: A surplus sharing characterization, complete and partial orderings. In: de Swart, H. (Ed.), Logic, Game Theory and Social Choice. University Press, Tilburg, pp. 427–441.

Appendices

Appendix A: The anthropometric achievements indices of the lower tail of the distribution

In this section we derive the achievement indices for the case of a bounded health indicator x that is in positive association with well-being. The related inequality indices are easily derived, and accordingly are not discussed here. In this context, $x_i - a$ denotes a health attainment, while $m - x_i$ is the health shortfall associated with x.

Let $W^X: D_{am} \to \mathbb{R}$ denote a social welfare function defined on a distribution of health attainments. Let W^X satisfy axioms of additivity, monotonicity, equality preference and scale invariance on D_{am} . By analogy with the discussion of Section 2, these standard axioms restrict the choice of a function $\phi(x-a)$ that is increasing on D_{am} to the family $u_\alpha(x-a)$ of power functions:

$$u_\alpha(x-a) = \begin{cases} \dfrac{(x-a)^{1-\alpha}}{1-\alpha}, & \alpha > 0, \alpha \neq 1 \\ \ln(x-a), & \alpha = 1 \end{cases} \tag{A.1}$$

Let $\hat{x}_R^a \in D_{am}$ denote the achievement index in the distribution X such that $u_\alpha(\hat{x}_R^a - a) = W_\alpha^X\left(x_1 - a, \dots, x_{n_2} - a\right)$. We obtain the following expression:

$$\hat{x}_R^a(X;\ \alpha) = \begin{cases} a + \left(\dfrac{1}{n_2}\displaystyle\sum_{i=1}^{n_2}(x_i - a)^{1-\alpha}\right)^{1/(1-\alpha)}, & \alpha > 0, \alpha \neq 1 \\ a + \exp\left(\dfrac{1}{n_2}\displaystyle\sum_{i=1}^{n_2}\ln(x_i - a)\right), & \alpha = 1 \end{cases} \tag{A.2}$$

For a social welfare function defined on the distribution of shortfalls, the associated utility function is of the form $u_\eta(m-x)=\frac{-(m-x)^{1-\eta}}{1-\eta}$ where $\eta<0$. The associated anthropometric achievement index $\hat{x}_R^s$ is given by the following expression:

$$\hat{x}_R^s(X,\eta)=m-\left(\frac{1}{n_2}\sum_{i=1}^{n_2}(m-x_i)^{1-\eta}\right)^{\frac{1}{1-\eta}}, \qquad \eta<0 \tag{A.3}$$

To derive the anthropometric absolute achievement index in relation to the distribution of health attainment on D_{am}, we similarly substitute a translation invariance axiom for the scale invariance axiom, and work with the family of exponential functions $u_\zeta(x-a)=1-\exp\big(-\zeta(x-a)\big)$, where $\zeta>0$. Accordingly, the equally distributed equivalent value $\hat{x}_A^a$ is of the form

$$\hat{x}_A^a(X;\zeta)=a-\frac{1}{\zeta}\ln\left(\frac{1}{n_2}\sum_{i=1}^{n_2}\exp\big(-\zeta(x_i-a)\big)\right) \tag{A.4}$$

For the distribution of shortfalls on D_{am}, the utility function associated with the absolute approach is of the from $u_\rho(m-x)=1-e^{-\rho(m-x)}$ where $\rho<0$. The associated anthropometric absolute achievement index is given by the following expression

$$\hat{x}_A^s(X;\rho)=m+\frac{1}{\rho}\ln\left(\frac{1}{n_2}\sum_{i=1}^{n_2}e^{-\rho(m-x_i)}\right), \qquad \rho<0 \tag{A.5}$$

Appendix B: Proofs of the main results

Proof of Proposition 1: Because, by assumption, $u()$ is a strictly decreasing function, it follows, that u^{-1} exists, and is strictly decreasing on the interval $[u(b^0),u(m^0)]$. Let $t:=\frac{1}{n_1}\sum_{i=1}^{n_1}u(y_i)$ be an element of the interval $[u(b^0),u(m^0)]$, and define the function $h(y_1,\ldots,y_{n_1}):=u^{-1}\left(\frac{1}{n_1}\sum_{i=1}^{n_1}u(y_i)\right)=u^{-1}(t)$, so that $\hat{y}=h(y_1,\ldots,y_{n_1})$.

Our next task is to show that $h(y_1, \ldots, y_{n_1})$ is a Schur-convex function. Because $u()$ is decreasing and concave, while $u^{-1}()$ is a decreasing function, it follows from result B1 (viii) of Marshall et al. (2011, p. 89) that $\hat{y} = h(y_1, \ldots, y_{n_1})$ is a Schur-convex function, that is a function decreasing in Pigou-Dalton transfers.

Because $u()$ is decreasing and concave, we have furthermore $\hat{y} = c\bar{y}$, where $c \geq 1$. The inequality $m \leq \bar{y}$ on the other hand holds because $Y \in D_{mb}$. ■

For the proofs of Propositions 4 and 5, we make use of the following preliminary result.

Lemma A *For all* $0 \leq \tau \leq 1$ *and* $0 \leq \theta \leq 1$, *and all* $Y \in D_{mb}$,

$$\hat{S}(Y; \theta, \tau) = g(\mu_Z\,; \theta, \tau, m, b) \cdot \hat{Z}(Y; \theta, \tau) \tag{B.1}$$

where

$$g(\mu_Z; \theta, \tau, m, b) = -\left(\frac{\theta\mu_Z + (1-\theta)}{\theta(b-m) + (1-\theta) - \theta\mu_Z}\right)^{\tau} \tag{B.2}$$

Furthermore, for all $Y \in D_{mb}$, *the following results hold:* (i) $g(\mu_z; 0, 0, m, b) = -1$, (ii) $g(\mu_z; \theta, 0, m, b) = -1$ *for all* $\theta \in (0,1]$, (iii) $g(\mu_z; 0, \tau, m, b) = -1$ *for all* $\tau \in (0,1]$, (iv) $\hat{S}(Y; 0,0) = -\hat{Z}(Y; 0,0)$, (v) $\hat{S}(Y; \theta, 0) = -\hat{Z}(Y; \theta, 0)$ *for all* $\theta \in (0,1]$ *and*

$$\hat{S}(Y; 0, \tau) = -\hat{Z}(Y; 0, \tau) \qquad \forall\, \tau \in (0,1] \tag{B.3}$$

Proof By definition, for a shortfall vector S,

$$\mu_S = \mu_Y - m \tag{B.4}$$

and for an attainment vector Z,

$$\mu_Z = b - \mu_Y \tag{B.5}$$

That is, $S - \mu_S \iota_{n_1} = S - (\mu_Y - m)\iota_{n_1} = Y - \mu_Y \iota_{n_1}$ and therefore $S - \mu_S \iota_{n_1} = -(Z - \mu_Z \iota_{n_1})$.

Using Eq. 15, we next express $d(S; \theta, \tau)$ as a function of b, m and μ_Z. Firstly, we have, $d(S; \theta, \tau) = [\,\theta\, \mu_s + (1-\theta)]^{\tau}$. Using Eq. B.4 and Eq. B.5, we have

$$d(S;\theta,\tau) = [\theta(b-m) - \theta\mu_Z + (1-\theta)]^\tau \tag{B.6}$$

Therefore, from Eq. 19, Eq. B.4, Eq. B.5 and Eq. B.6, we have that

$$\hat{S}(Y;\theta,\tau) = -\frac{(Z-\mu_Z\iota_{n_1})}{[\theta(b-m) - \theta\mu_Z + (1-\theta)]^\tau} \tag{B.7}$$

Recalling from Eq. 18 that $\hat{Z}(Y;\theta,\tau) = -\left(\frac{Z-\mu_Z\iota_{n_1}}{d(Z;\theta,\tau)}\right)$, we may define the function

$$g(\mu_Z;\theta,\tau,m,b) = -\left(\frac{\theta\mu_Z + (1-\theta)}{\theta(b-m) + (1-\theta) - \theta\mu_Z}\right)^\tau$$

and arrive at $\hat{S}(Y;\theta,\tau) = g(\mu_Z;\theta,\tau,m,b)\cdot\hat{Z}(Y;\theta,\tau)$ as required. The remaining results follow from the evaluation of the function g for specific values of θ and τ. ■

Proof of Proposition 4 We introduce the following compact notation: for $i = A, B$ we let $\hat{Z}^i(\theta,\tau)$ denote the vector $\hat{Z}\left(Y^i;\theta,\tau\right)$, and similarly, $\hat{S}^i(\theta,\tau)$ denotes the vector $\hat{S}\left(Y^i;\theta,\tau\right)$.

Consider point (a): we must show that if $\hat{Z}^A \sim_{\tau,\theta} \hat{Z}^B$ and $\tau\theta = 0$, then it is also the case that $\hat{S}^A \sim_{\tau,\theta} \hat{S}^B$. If $\theta\tau = 0$, we have either of three cases $\theta = \tau = 0$, $\theta \neq 0$ and $\tau = 0$, or $\theta = 0$ and $\tau \neq 0$. Consider any of these three cases, say the last one, then using Eq. B.3

$$\hat{Z}^A(0,\tau) \sim_{\tau,\theta} \hat{Z}^B(0,\tau) \Leftrightarrow \hat{Z}^A(0,\tau)\downarrow = \hat{Z}^B(0,\tau)\downarrow$$

$$\Leftrightarrow -\left[\hat{Z}^A(0,\tau)\downarrow\right] = -\left[\hat{Z}^B(0,\tau)\downarrow\right] \Leftrightarrow \hat{S}^A(0,\tau)\uparrow = \hat{S}^B(0,\tau)\uparrow$$

$$\Leftrightarrow \hat{S}^A(0,\tau) \sim_{\tau,\theta} \hat{S}^B(0,\tau).$$

The argument is identical when considering the two cases $\theta = \tau = 0$, and $\theta \neq 0$, $\tau = 0$.

Turning to (b), we now examine the case where $\hat{Z}^A(\theta,\tau)\downarrow = \hat{Z}^B(\theta,\tau)\downarrow$ with both $\theta \neq 0$ and $\tau \neq 0$. Define $\hat{Z}(\theta,\tau) := \hat{Z}^A(\theta,\tau)\downarrow = \hat{Z}^B(\theta,\tau)\downarrow$. From Eq. B.1 therefore,

$$\hat{S}^A(\theta,\tau) \succ_{\tau,\theta} \hat{S}^B(\theta,\tau) \Leftrightarrow g\left(\mu_{Z^A};\theta,\tau,m,b\right)\cdot\hat{Z}(\theta,\tau) \succ_{\tau,\theta} g(\mu_{Z^B};\theta,\tau,m,b)\cdot\hat{Z}(\theta,\tau)$$

$$\Leftrightarrow g\left(\mu_{Z^A};\theta,\tau,m,b\right) > g(\mu_{Z^B};\theta,\tau,m,b) \Leftrightarrow \mu_{Z^A} < \mu_{Z^B}$$

as the function g is decreasing in its first argument. ■

Proof of Proposition 5 The result follows from Proposition 4, and accordingly the proof is omitted.

Proof of Proposition 6 Starting from Eq. 20, we have the following sequence of equivalent statements:

$$Z^B \prec_w Z^A \Leftrightarrow \textstyle\sum_{i=1}^{k}(b - y_{[i]}^B) \leq \sum_{i=1}^{k}(b - y_{[i]}^A) \qquad k = 1,2,\dots,n_1$$

$$\Leftrightarrow \sum_{i=1}^{k}(y_{[i]}^A - m) \leq \sum_{i=1}^{k}(y_{[i]}^B - m), \qquad k = 1,2,\dots,n_1$$

$$\Leftrightarrow S^A \prec_{SM} S^B,$$

where the last equivalence follows from Eq. 21. ■

Proof of Proposition 7 The proof consists in showing that for any distribution $Y \in D_{mb}$, and for any value $\kappa > 0$ the following identity holds: $\hat{y}_A^a(Y;\, \kappa) = \hat{y}_A^s(Y; -\kappa)$. Add and subtract m in Eq. 11 to write:

$$\hat{y}_A^a(Y;\, \kappa\,) = b + \tfrac{1}{\kappa}\ln\left(\tfrac{1}{n_1}\textstyle\sum_{i=1}^{n_1}\exp\big(\kappa(y_i - m) - \kappa(b - m)\big)\right),$$

equivalently,

$$\hat{y}_A^a(Y;\, \kappa\,) = b + \tfrac{1}{\kappa}\ln\left(\tfrac{1}{n_1}\textstyle\sum_{i=1}^{n_1}\exp\big(\kappa(y_i - m)\big)\exp\big(-\kappa(b - m)\big)\right)$$

Factoring the constant term out of the sum, we obtain

$$\hat{y}_A^a(Y;\, \kappa\,) = b + \tfrac{1}{\kappa}\ln\left(\exp\big(-\kappa(b - m)\big)\tfrac{1}{n_1}\textstyle\sum_{i=1}^{n_1}\exp\big(\kappa(y_i - m)\big)\right)$$

that is,

$$\hat{y}_A^a(Y;\, \kappa) = b + \frac{1}{\kappa}\ln\Big(\exp\big(-\kappa(b - m)\big)\Big) + \frac{1}{\kappa}\ln\left(\frac{1}{n_1}\sum_{i=1}^{n_1}\exp\big(\kappa(y_i - m)\big)\right).$$

For $\delta = -\kappa$, we then have that $\hat{y}_A^a(Y;\, \kappa) = m - \tfrac{1}{\delta}\ln\left(\tfrac{1}{n_1}\sum_{i=1}^{n_1}\exp\big(-\delta(y_i - m)\big)\right)$. That is, we have shown, as required, that $\hat{y}_A^a(Y;\, \kappa) = \hat{y}_A^s(Y, -\kappa)$. ■

Tables

Table 1.A: Anthropometric relative achievement and inequality of health attainments in five Arab countries: the upper tail of the BMI distribution ($m = 24.9$, $b = 60$)

		Countries	Egypt	Jordan	Comoros	Yemen	Morocco
		Sample sizes, n_1	5226	6336	1926	5666	6238
		Mean, $\bar{y}$	32.40	31.02	29.33	29.21	28.82
		Variance	32.62	24.73	17.09	15.25	12.82
Inequality-aversion parameter (β)	0.5	Achievement index, $\hat{y}_R^a$	32.76	31.27	29.49	29.35	28.93
		AKS inequality index, I_R^a	0.0131	0.0087	0.0054	0.0047	0.0037
	1	Achievement index, $\hat{y}_R^a$	33.20	31.56	29.68	29.51	29.06
		AKS inequality index, I_R^a	0.0291	0.0186	0.0116	0.0099	0.0078
	3	Achievement index, $\hat{y}_R^a$	40.86	33.60	30.96	30.46	29.73
		AKS inequality index, I_R^a	0.3067	0.0890	0.0531	0.0408	0.0294

Table 1.B: Anthropometric relative achievement and inequality of health shortfalls in five Arab countries: the upper tail of the BMI distribution ($m = 24.9$, $b = 60$)

		Countries	Egypt	Jordan	Comoros	Yemen	Morocco
Inequality-aversion parameter (γ)	-0.5	Achievement index, $\hat{y}_R^s$	33.4072	31.9410	30.1528	29.9775	29.5292
		AKS inequality index, I_R^s	0.1186	0.1310	0.1580	0.1517	0.1541
	-1	Achievement index, $\hat{y}_R^s$	34.3254	32.7847	30.9566	30.7138	30.2063
		AKS inequality index, I_R^s	0.2044	0.2240	0.2690	0.2591	0.2620
	-3	Achievement index, $\hat{y}_R^s$	37.4575	35.6684	33.9360	33.4364	32.6385
		AKS inequality index, I_R^s	0.4029	0.4318	0.5100	0.4954	0.4940

Table 2. Anthropometric absolute achievement and inequality of health attainments in five Arab countries: the upper tail of the BMI distribution ($m = 24.9$, $b = 60$)

Inequality-aversion parameter (κ)		Countries	Egypt	Jordan	Comoros	Yemen	Morocco
	0.5	Achievement index, $\hat{y}_A^a$	45.87	42.97	40.84	39.10	36.78
		Kolm inequality index, I_A^a	13.47	11.95	11.51	09.86	07.97
	1	Achievement index, $\hat{y}_A^a$	51.69	49.15	46.48	44.89	42.85
		Kolm inequality index, I_A^a	19.29	18.13	17.15	15.69	14.03
	3	Achievement index, $\hat{y}_A^a$	55.33	52.75	49.57	48.39	46.63
		Kolm inequality index, I_A^a	22.93	21.73	21.24	19.18	17.82

Figures

Figure 1: Absolute Lorenz curves: the upper tail of the BMI distribution in five Arab countries (b=60 and m=24.9)

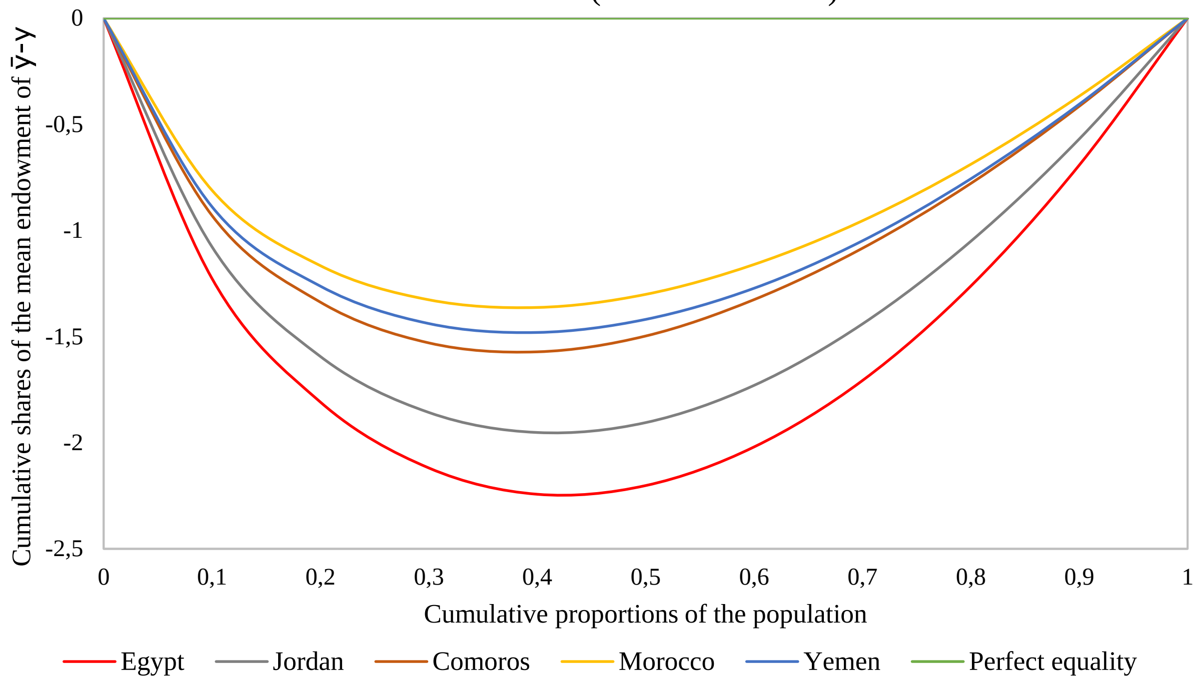


Figure 2: Generalized Lorenz curves: the upper tail of the BMI distribution in five Arab Countries (b=60 and m=24.9)

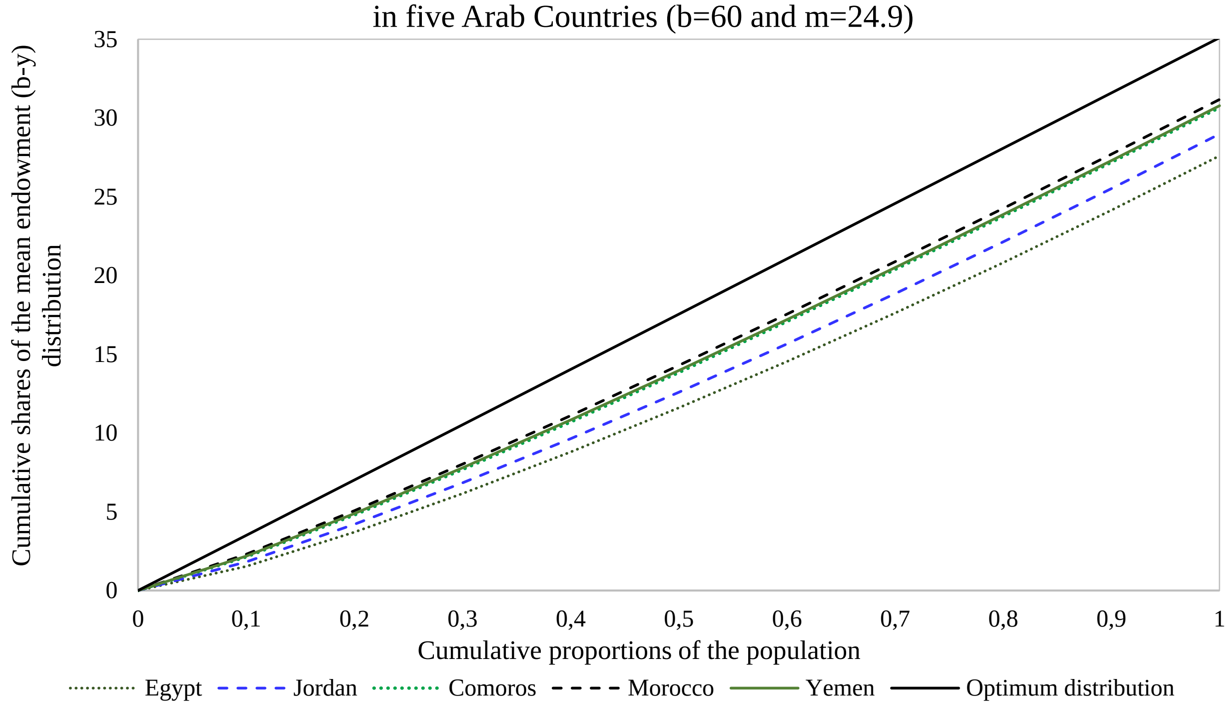


Figure 3: Generalized Lorenz curves: the lower tail of the BMI distribution in five Arab Countries (a = 10 and m = 24.9)

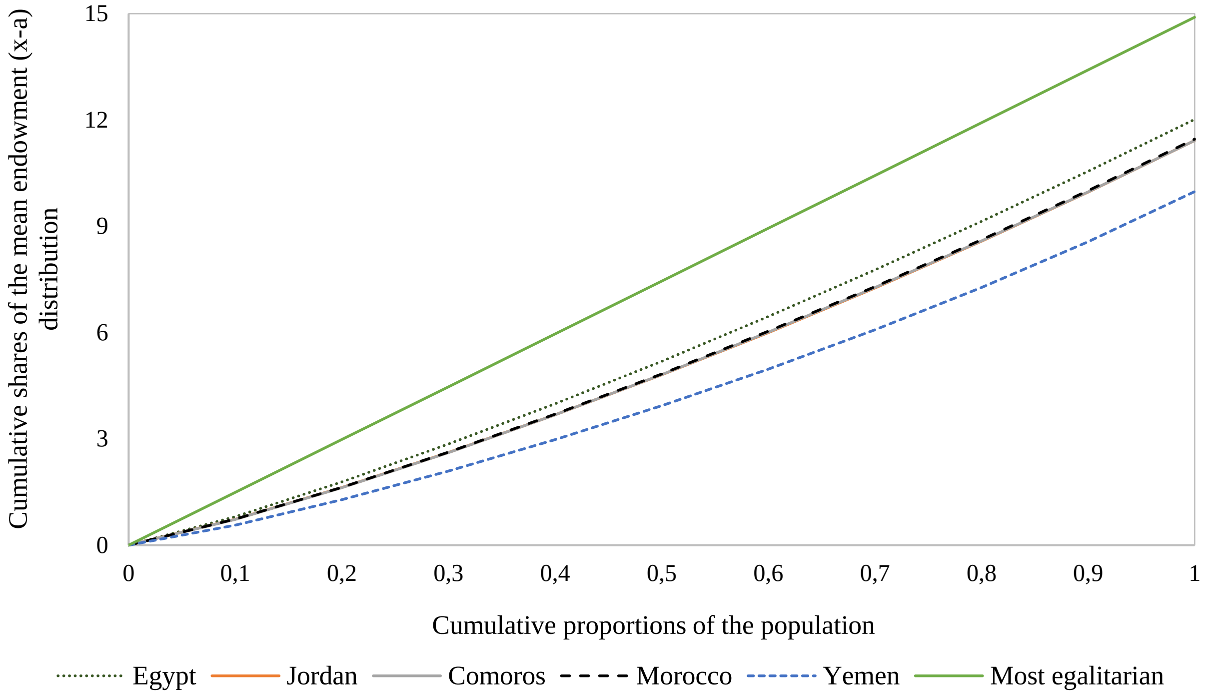
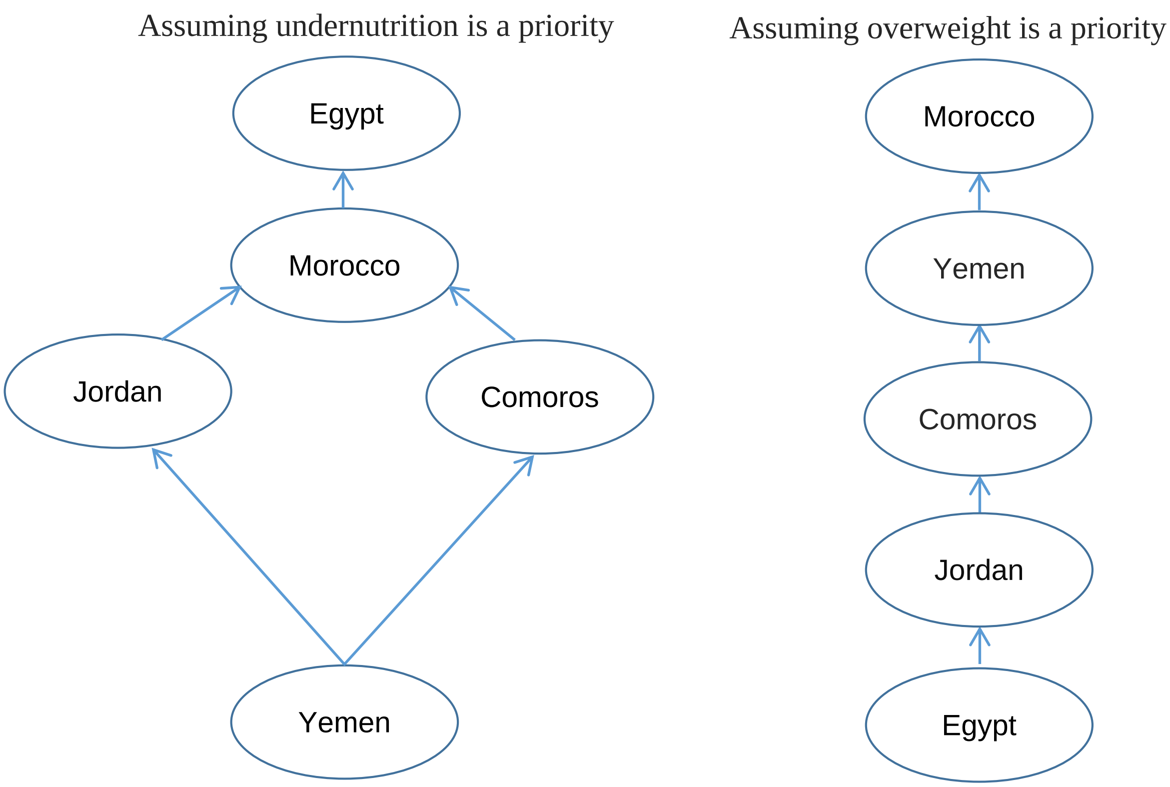
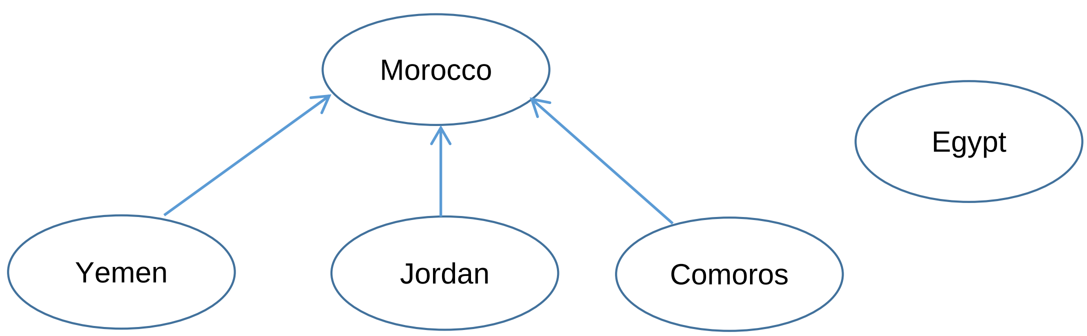


Figure 4: Hasse diagram of the lexicographic order *



* An arrow pointed from country A to country B indicates that social welfare is higher in country B according to the lexicographic order

Figure 5: Hasse diagram of the product order *



* An arrow pointed from country A to country B indicates that social welfare is higher in country B according to the product order.