Málaga Economic Theory
Research Center
Working Papers

The problem of aggregating experts' opinions to select the winner of a competition

Pablo Amorós

WP 2017-4
November 2017

Departamento de Teoría e Historia Económica
Facultad de Ciencias Económicas y Empresariales
Universidad de Málaga
ISSN 1989-6908

The problem of aggregating experts' opinions to select the winner of a competition*

Pablo Amorós†

November 30, 2017

Abstract

The honest opinions of a group of experts must be aggregated to determine the deserving winner of a competition. The aggregation procedure is majoritarian if, whenever a majority of experts honestly believe that a contestant is the best one, then that contestant is considered the deserving winner. The fact that an expert believes that a contestant is the best one does not necessarily imply that she wants this contestant to win as, for example, she might be biased in favor of some other contestant. Then, we have to design a mechanism that implements the deserving winner. We show that, if the aggregation procedure is majoritarian, such a mechanism exists only if the experts are totally impartial. This impossibility result is very strong as it does not depend on the equilibrium concept considered. Moreover, the result still holds if we replace majoritarianism by anonymity and other reasonable property called respect for the jury. The impossibility result is even stronger if we focus on Nash implementation: no majoritarian aggregation procedure can be Nash implemented even if the experts are totally impartial.

Key Words: mechanism design; aggregation of experts' opinions; jury.

J.E.L. Classification Numbers: C72, D71, D78.

*Financial assistance from Ministerio de Economía y Competitividad under project ECO2014-53767-P is gratefully acknowledged.

†Universidad de Málaga, Campus El Ejido, E-29013, Málaga, Spain; tel. +34 95 213 1245, fax: +34 95 213 1299; e-mail: pag@uma.es.

1 Introduction

Consider the problem of a group of experts that must choose the winner of a competition among a group of contestants. No contestant is unequivocally better for all experts. Different experts may have different opinions about who is the best contestant. The opinions of the experts must be aggregated to determine which contestant is socially considered to be the deserving winner of the competition. Experts have preferences over the contestants that may depend on their own opinions about who is best contestant. However, the fact that an expert believes that a contestant is the best one does not necessarily imply that she wants this contestant to win. For example, an expert might be biased in favor of some contestant x and always prefer x to win the competition, independently on whether she believes x is the best one or not. An example of this type of problems is the Olympic host city election. The candidate cities are the contestants and the members of the International Olympic Committee (IOC) are the experts. Different IOC members may have different opinions about which city is the best candidate. These opinions must be aggregated to determine which city is the deserving winner. The fact that an IOC member honestly believes that certain city is the best candidate does not necessarily imply that she wants this city to be selected. For political or corruption reasons, she might be biased in favor of another city. Other examples are the selection of Nobel laureates or the hiring process in a department.

It is important to differentiate between the process of aggregation of experts' honest opinions to determine the deserving winner, and the voting procedure used by the experts to decide the actual winner of the competition. The former reflects the objectives of the society, while the latter is the mechanism used to implement these objectives. The process of aggregation of experts' honest opinions can be represented by a social choice function (SCF) that selects, for each admissible profile of opinions, the contestant who is socially considered to be the deserving winner. Because the experts may be biased, they may not want to reveal their opinions about who is the best contestant. For this reason, we have to design a mechanism that gives the incentives to the experts to always choose the deserving winner. For example, a mechanism could be the voting system used by the experts to choose the winner of the competition. Ideally, we are able to find a voting system such that, in equilibrium, the experts always choose the contestant who is socially considered to be the deserving winner according to their honest opinions.

When this happens, we say that the SCF is implementable.

In this paper, we focus on analyzing selection committees in which the opinions of all experts are equally important when determining the deserving winner. Based on this idea, a reasonable requirement that a SCF should satisfy is majoritarianism: whenever one contestant is honestly viewed as the best one by a majority of experts, then that contestant should be considered as deserving winner by the SCF. Majoritarianism is at the essence of the process of aggregation of expert opinions and it underlies as a requirement in many real-world problems. For a majoritarian SCF to be implementable, there must be certain limits in the degree of bias of the experts. To understand this notice that, for example, if all the experts were biased in favor of the same contestant, they would always manage to make her win the competition, even if a majority of them honestly believe that the best contestant is another one (and regardless of the mechanism that they use). Our aim is to analyze conditions on the bias of the experts so that some majoritarian SCF exists that can be implemented. In order to classify the degree of bias of an expert, we use the concept of being impartial with respect to a pair of contestants. We say that an expert is impartial with respect to a pair of contestants if, whenever she believes that one of the two contestants is the best contestant of the competition, then she prefers that contestant to the other.

Amorós (2013) made the simplifying assumption that all experts have always the same opinion about who is the best contestant (although different experts may have different biases in their preferences). Clearly, in this case the only reasonable SCF is the one that always selects the contestant who is viewed as the best one by all experts. A necessary condition for the implementation of this SCF in any ordinal equilibrium concept is that, for each pair of contestants, at least one expert must be impartial with respect to them. This condition, called “minimal impartiality”, is also sufficient for the implementability of the SCF when the equilibrium concept considered is Nash equilibrium.

In the present paper, we analyze the more interesting case where different experts may have different opinions about who is the best contestant. Unlike what happens when all experts have the same opinion, in this case there is no single, trivial way to aggregate their opinions to determine the deserving winner. Then, the question we try to answer is: what are the constraints on the bias of the experts so that there is at least one SCF that satisfies majoritarianism and that can be implemented in some equilibrium concept?

Note that this question is very general, since we do not study any specific SCF or focus on any particular equilibrium concept.

Unfortunately, the requirements for the existence of a majoritarian SCF that can be implemented are very demanding. We show that, if the number of experts is odd, then no majoritarian SCF can be implemented in any equilibrium concept unless all experts are impartial with respect to each pair of contestants (Theorem 1). If the number of experts is even, the condition is weaker, but still very demanding: no majoritarian SCF can be implemented in any equilibrium concept unless all experts but possibly one are impartial with respect to each pair of contestants (Theorem 2). We call total impartiality and quasi-impartiality to the conditions stated in Theorems 1 and 2, respectively. The intuition of these results is as follows. An expert is said to be decisive at a pair of contestants x and y if, for some fixed opinions of the rest of experts, the deserving winner selected by the SCF changes when this expert opinion changes from believing that the best contestant is x to believing that the best contestant is y. It turns out that, if a SCF is implementable in some equilibrium concept, then each expert must be impartial with respect to each pair of contestants in which she is decisive. Otherwise, we could find two profiles of experts' opinions for which the deserving winners selected by the SCF were different and, despite this, the preference relations of each expert were the same in both situations (which makes the implementation of the SCF impossible, whatever equilibrium concept is used). If a SCF is majoritarian and the number of experts is odd, every expert is decisive at every pair of contestants and then, if the SCF is implementable, she must be impartial with respect to them. Similarly, if the number of experts is even, all experts but possible one are decisive for each pair of contestants and then, if the SCF is implementable, they must be impartial with respect to them.

In most cases it is unrealistic to believe that the experts are totally impartial (or quasi-impartial) as, for example, some of them have friends or enemies among the contestants. Therefore, Theorems 1 and 2 can be interpreted as showing that no majoritarian SCF can be implemented in these cases. These impossibility results are very consistent. First, they do not depend on the equilibrium concept considered. Second, for the results to hold, it is sufficient that there are two different opinions among the experts about who is the best contestant. Third, the results still hold if we replace majoritarianism by two other reasonable properties: respect for the jury (the contestant selected by the SCF must be considered as the best one by at least one expert) and anonymity (changing the names of the experts with each opinion would not change the contestant socially considered to be the deserving winner). In fact, in this case, total impartiality is a necessary condition for implementation both when the number of experts is odd or even (Theorem 3).

As strong as they are, the fulfillment of the necessary conditions stated in Theorems 1 and 2 does not necessarily guarantee the existence a majoritarian SCF that is implementable. Whether these conditions are sufficient or not may depend on the equilibrium concept considered. We study the particular case of Nash equilibrium and show that the problem of finding a majoritarian SCF that can be implemented is even more difficult than the previous results indicated. If there are at most two different opinions among the experts and the number of experts is odd and greater than or equal to three, then the necessary condition stated in Theorem 1 is sufficient when the equilibrium concept is Nash equilibrium. In fact, in this case every majoritarian SCF is Nash implementable (Theorem 4). If there are at most two different opinions among the experts and the number of experts is even, the necessary condition stated in Theorem 2 is not sufficient. In this case, to guarantee the existence of a majoritarian SCF that is implementable in Nash equilibrium, we need that all experts but possible one are impartial with respect to all pairs of contestants (Theorem 5). This condition, that we call strict-quasi-impartiality, is stronger than quasi impartiality since now the same experts must be impartial with respect to all experts. Finally, if there can be three or more different opinions among the experts, then no majoritarian SCF can be implemented in Nash equilibrium, regardless of whether total impartiality or any other restriction on the degree of bias of the experts is fulfilled (Theorem 6).

Related literature

There are a few predecessors to this paper studying models in which all experts have the same opinion. Amorós (2013) studies the case where there can be more than one winner in the competition and all experts agree on who are the best contestants. In this case, there is no need to aggregate the experts' opinions and the only reasonable SCF is that which selects the contestants who are viewed as the best ones by all experts. He shows that minimal impartiality is a necessary condition for the implementability of this SCF in any equilibrium concept, and a sufficient condition for its implementability in Nash equilibrium. Amorós (2009) analyzes a model where the experts must choose a full ranking of the contestants under the assumption that they all agree on which the true ranking is. As in the previous paper, there is no need to aggregate different opinions and the only reasonable SCF is that which selects the true ranking observed by all experts. The necessary and sufficient condition for the Nash implementability of this SCF is very similar to minimal impartiality.

The literature on the Condorcet Jury Theorem also deals with the problem of juries whose members are strategic (e.g., Austen-Smith and Banks, 1996; Dugan and Martinelli, 2001; Feddersen and Pesendorfer; 1998, McLennan, 1998). These papers study the case where the experts must choose between two alternatives and agree on the overall objective, but on the basis of differential information, they may disagree on which alternative is the best one. There are several differences between this literature and our approach. In our model, no contestant is unequivocally better for all experts, but it is the experts' owns opinions that determine who is the deserving winner. Moreover, in this literature, the incentive to vote strategically arises because an expert's vote only matters when she is pivotal and because the information possessed by other experts is relevant for an expert's decision, not because the experts are biased.

Another related strand in the literature is the theory of judgement aggregation (e.g., Pettit, 2001; List and Pettit, 2002). This literature analyzes how a group of experts can make consistent collective judgements on a set of propositions on the basis of the experts' individual judgments on them. In our model there is no problem of inconsistent judgements because experts do not have to judge different propositions. However, both the judgement aggregation approach and our approach point out certain weaknesses of majority processes. The former shows that majority voting fails to guarantee consistent collective judgements, while the latter shows that majoritarian aggregation procedures fail to be implementable.

Some papers in the literature deal with aggregation of opinions with uncertainty. For example, Crès et al. (2011) analyze the problem of aggregating experts' beliefs when they adopt the decision maker's utility function but have different opinions about the prior probabilities (see also Gajdos and Vergnaud, 2013). In our work, however, experts have no uncertainty, but simply have different opinions about who is the best contestant.

Finally, the present paper is also connected with the literature on information transmission between informed experts and an uninformed decision maker (e.g., Gerardi et al., 2009; Krishna and Morgan, 2001; Wolinsky, 2002). These papers analyze the problem where a group of experts are called to advise a decision maker who has different preferences. Krishna and Morgan (2001) study the case in which two informed and biased experts offer advice to a decision maker and show that, if both experts are biased in the same direction, then there is no equilibrium in which full revelation occurs (this situation bears some resemblance to the case in which all experts want to favor the same contestant in our model). In the model analyzed by Wolinsky (2002), the experts share a similar bias relative to the decision maker, but they have different pieces of information so that their reports cannot be confronted. He shows that (if the decision maker can commit to a mechanism) it is sometimes possible to elicit more information than the experts would like to reveal. Gerardi et al. (2009) investigate how the decision maker can extract the information by distorting the decisions that will be taken.

The reminder of this paper is organized as follows. In Section 2 we explain our basic model. In Section 3 we present the necessary conditions for the existence of a majoritarian SCF that can be implemented in some equilibrium concept. In Section 4, we analyze the particular case where the equilibrium concept is Nash equilibrium. In Section 5, we give our conclusions. The Appendix provides proofs of some of the results.

2 The model

A jury composed of a group of experts $E = \{1, 2, ...\}$ must choose the winner of a competition among a group of contestants $N = \{a, b, ...\}$. The general elements of E are denoted i, j, etc. and the general elements of N are denoted x, y, etc. Different experts may have different opinions about who is the best contestant. For each expert i, let $w_i \in N$ be the contestant who i thinks is the best one. Let $w = (w_i)_{i \in E} \in N^{|E|}$ denote the profile of experts' opinions about who is the best contestant.

The number of different opinions among experts may be limited. We call this notion the **limit of opinions** and represent it with an integer δ such that $1 \leq \delta \leq \min\{|E|, |N|\}$. For each profile of experts' opinions $w \in N^{|E|}$, let $N_w = \{x \in N : x = w_i \text{ for some } i \in E\}$ be the set of contestants who are perceived as the best one by some expert in w. Given δ, we say that $w \in N^{|E|}$ is an **admissible profile of experts' opinions** if $|N_w| \leq \delta$, that is, if there are at most δ different opinions within the group of experts about who is the best contestant. If $\delta = \min\{|E|, |N|\}$ then we admit the possibility that all experts disagree about who is the best contestant. The idea behind the case $\delta < \min\{|E|, |N|\}$ is that, although different experts may have different opinions about who is the best contestant, these opinions can not be "too different". In the extreme case where $\delta = 1$, all experts have always the same opinion about who is the best contestant. Let $W(\delta) = \{w \in N^{|E|} : |N_w| \leq \delta\}$ be the set of admissible profiles of experts' opinions given that limit of opinions is δ.

Example 1 *Suppose* $E = \{1, 2, 3\}$ *and* $N = \{a, b, c\}$*. Then* $W(3)$ *consists of the* 27 *profiles of opinions shown in Table 1 (xyz is a profile of opinions* w *where* $w_1 = x$*,* $w_2 = y$*, and* $w_3 = z$*). The set* $W(2)$ *consists of* 21 *profiles of opinions (the same as in* $W(3)$ *except abc, acb, bac, bca, cab, and cba). The set* $W(1)$ *has only* 3 *admissible profiles of opinions: aaa, bbb, and ccc.*

$W(3)$

aaa	*baa*	*caa*
aab	*bab*	*cab*
aac	*bac*	*cac*
aba	*bba*	*cba*
abb	*bbb*	*cbb*
abc	*bbc*	*cbc*
aca	*bca*	*cca*
acb	*bcb*	*ccb*
acc	*bcc*	*ccc*

$W(2)$

aaa	*baa*	*caa*
aab	*bab*	*cac*
aac	*bba*	*cbb*
aba	*bbb*	*cbc*
abb	*bbc*	*cca*
aca	*bcb*	*ccb*
acc	*bcc*	*ccc*

$W(1)$

aaa	*bbb*	*ccc*

Table 1 Admissible profiles of opinions when $E = \{1, 2, 3\}$ and $N = \{a, b, c\}$ depending on the limit of opinions δ.

The opinions of the experts must be aggregated to determine who is socially considered to be the deserving winner of the competition. The aggregation process is represented by a **social choice function** (SCF). A SCF is a mapping from the set of admissible profiles of experts' opinions into the set of contestants, $F : W(\delta) \to N$. Given a profile of experts' opinions w, $F(w)$ is the contestant socially considered as deserving winner. Let $\mathcal{F}(\delta)$ denote the class of all possible SCFs given the limit of opinions δ.

It seems reasonable that, whenever the same contestant is viewed as the best one by more than half of the experts, then that contestant should be considered as deserving winner. We say that a SCF is majoritarian if it satisfies this property. For each $w \in W(\delta)$ and $x \in N$, let $E_w^x = \{i \in E : w_i = x\}$ be the set of experts who think x is the best contestant in w.

Definition 1 *A SCF $F \in \mathcal{F}(\delta)$ is* ***majoritarian*** *if, whenever $w \in W(\delta)$ is such that there is $x \in N$ with $|E_w^x| > \frac{|E|}{2}$, then $F(w) = x$. Let $\mathcal{F}^M(\delta) \subset \mathcal{F}(\delta)$ denote the set of all majoritarian SCFs.*[1]

Experts have preferences over the contestants that may depend on their own opinions about who is the best contestant. Let $\Re$ denote the class of all complete, reflexive, and transitive preference relations over N. Each expert i has a preference function $R_i : N \longrightarrow \Re$ that associates with each possible opinion of i about who is the best contestant, $w_i \in N$, a preference relation $R_i(w_i) \in \Re$. The fact that an expert i believes w_i is the best contestant does not necessarily imply that w_i is her most preferred contestant. For example, i might be biased in favor some contestant x and always prefer x to win the competition, independently of whether or not she believes x is the best contestant. Let $P_i(w_i)$ denote the strict part of $R_i(w_i)$. Let $\mathcal{R}$ denote the class of all possible preference functions.

Example 2 *In Table 2 we show an example of preference function for the case $E = \{1, 2, 3\}$ and $N = \{a, b, c\}$. Contestants ranked higher in the table are strictly preferred to those ranked lower.*

$R_i : N \longrightarrow \Re$			
$w_i =$	a	b	c
	a	a	a
Preferences	bc	b	c
		c	b

Table 2 Example of preference function when E= $\{1, 2, 3\}$ and $N = \{a, b, c\}$.

Let 2_2^N denote the set of all possible pairs of contestants. We say that an expert i is **impartial** with respect to a pair of contestants if, whenever she believes that one of the two contestants is the best contestant of the

[1]More generally, one could think that the opinion of an expert is a complete ranking of the contestants (from the best to the worst) and that a SCF is a mapping from the set of admissible profiles of rankings of the experts into the set of contestants,. In this case, a SCF would be majoritarian if, whenever a majority of the experts ranked the same contestant in the first position, then that contestant would be selected. Because, in order to know whether or not a SCF is majoritarian, we only take into account the experts' opinions on who is the best contestant, then all the results in the present paper continue to hold in this more general setting.

competition, she prefers that contestant to the other. Each expert i is characterized by a **set of pairs of contestants with respect to whom she is impartial**, $I_i \subset 2_2^N$.

Definition 2 *A preference function $R_i \in \mathcal{R}$ is* ***admissible*** *for expert i at $I_i \subset 2_2^N$ if, for each $xy \in I_i$, (i) whenever $x = w_i$, then $x\ P_i(w_i)\ y$, and (ii) whenever $y = w_i$, then $y\ P_i(w_i)\ x$.*[2]

Let $\mathcal{R}(I_i)$ be the class of all preference functions that are admissible for i at I_i. A **jury configuration** is a profile of sets of pairs of contestants with respect to whom the experts are impartial, $I = (I_i)_{i \in E}$. A profile of preference functions $R = (R_i)_{i \in E}$ is admissible at jury configuration I if $R_i \in \mathcal{R}(I_i)$ for every $i \in E$. Abusing notation, we write $\mathcal{R}(I) \subset \mathcal{R}^{|E|}$ to denote the set of admissible profiles of preference functions at I. Given a jury configuration I and a pair of contestants xy, let $E_{xy}^I = \{i \in E : xy \in I_i\}$ be the set of experts who are impartial with respect to xy at I. We say that a jury configuration is minimally impartial if, for each pair of contestants, at least one expert is impartial with respect to them.

Definition 3 *A jury configuration I is* ***minimally impartial*** *if, for each $xy \in 2_2^N$, $\left|E_{xy}^I\right| \geq 1$.*

We say that a jury configuration is totally impartial (quasi-impartial) if all experts (but possibly one) are impartial with respect to each pair of contestants.

Definition 4 *A jury configuration I is* ***totally impartial*** *if, for each $xy \in 2_2^N$, $E_{xy}^I = E$.*

Definition 5 *A jury configuration I is* ***quasi-impartial*** *if, for each $xy \in 2_2^N$, $\left|E_{xy}^I\right| \geq |E| - 1$.*

Total impartiality is a very demanding condition. It requires that, for every admissible profile of preference functions $R \in \mathcal{R}(I)$ and every admissible profile of experts' opinions $w \in W(\delta)$, the most preferred contestant for each expert i is w_i, *i.e.*, $w_i\ P_i(w_i)\ x$ for every $x \in N \backslash \{w_i\}$. Note that, for example, this prevents the experts from having friends or enemies among

[2] For example, the preference function in Table 2 is admissible at $I_i = \{bc\}$, but it is not admissible at $I_i = \{ab\}$.

the contestants (*i.e.*, contestants to whom they always want to favor or contestants to whom they always want to harm). Quasi-impartiality is a weaker condition, but still very demanding.

A SCF represents the socially optimal way to aggregate the experts' opinions about who is the best contestant. However, w_i is privately observed by each expert i. Thus, we have to design a mechanism that implements the SCF. A **mechanism** is a pair $\Gamma = (M, g)$, where $M = \times_{i \in E} M_i$, M_i is a message space for expert i, and $g : M \to N$ is an outcome function. Given a jury configuration I and a limit of opinions δ, a **state** is a profile of admissible preference functions together with an admissible profile of expert's opinions, $(R, w) \in \mathcal{R}(I) \times W(\delta)$. Let $\mathcal{E}$ be a game theoretic equilibrium concept. For each mechanism Γ and each state (R, w), let $\mathcal{E}(\Gamma, R, w) \subset M$ denote the set of profiles of messages that are an $\mathcal{E}$-equilibrium of mechanism Γ at state (R, w). Let $\mathbb{E}$ be the class of all ordinal equilibrium concepts (*i.e.*, the class of equilibrium concepts $\mathcal{E}$ such that, for every $(R, w), (\hat{R}, \hat{w}) \in \mathcal{R}(I) \times W(\delta)$ with $R_i(w_i) = \hat{R}_i(\hat{w}_i)$ for every $i \in E$, we have $\mathcal{E}(\Gamma, R, w) = \mathcal{E}(\Gamma, \hat{R}, \hat{w})$).[3] For example, $m \in M$ is a **Nash equilibrium** of mechanism $\Gamma = (M, g)$ at state (R, w) if for each $i \in E$ and each $\hat{m}_i \in M_i$, $g(m)\ R_i(w_i)\ g(\hat{m}_i, m_{-i})$.

A mechanism implements a SCF in $\mathcal{E}$-equilibrium if, in every state, the contestant prescribed by the SCF is selected in equilibrium.

Definition 6 *Given an equilibrium concept $\mathcal{E} \in \mathbb{E}$, a jury configuration I, and a limit of opinions δ, a mechanism $\Gamma = (M, g)$* ***implements*** *a SCF $F \in \mathcal{F}(\delta)$ in $\mathcal{E}$-equilibrium, if, for each admissible state $(R, w) \in \mathcal{R}(I) \times W(\delta)$:*
(i) $\mathcal{E}(\Gamma, R, w) \neq \emptyset$, and
(ii) $m \in \mathcal{E}(\Gamma, R, w)$ if and only if $g(m) = F(w)$.

A SCF is implementable in $\mathcal{E}$-equilibrium (or it is $\mathcal{E}$-implementable) if a mechanism exists that implements it in $\mathcal{E}$-equilibrium.[4]

[3] For each $\mathcal{E} \in \mathbb{E}$, if no juror changes her preferences from state (R, w) to state $(\hat{R}, \hat{w})$, then the profiles of messages that constitute an $\mathcal{E}-$equilibrium are the same in both states.

[4] One can also use extensive form mechanisms to implement a SCF. An extensive form mechanism is a dynamic mechanism in which experts make choices sequentially. The definition of implementation can easily be extended to this type of mechanisms. Although, in general, the use of extensive form mechanisms facilitates the implementation problem, our results in the present paper still hold when we consider these mechanisms.

3 General results

We aim at studying what conditions must satisfy the jury configuration so that a majoritarian SCF exists that is implementable in some ordinal equilibrium concept. In the simplest case where the limit of opinions is one, the only majoritarian SCF is the one that selects the contestant who is viewed as best contestant by all experts. Amorós (2013) analyzed this case and showed that minimal impartiality is a necessary condition for the implementability of this SCF in any equilibrium concept, and a sufficient condition for its implementability in Nash equilibrium. In this paper, we analyze the more interesting case where $\delta > 1$, so that different experts may have different opinions about who is the best contestant.

We begin by stating two simple lemmas that will be useful in our analysis. The first one states that, if an expert i is not impartial with respect to a pair of contestants xy, then there exists an admissible preference function for i that is such that her preferences when she believes that the best contestant is x are the same than when she believes that the best contestant is y.

Lemma 1 *For every expert $i \in E$ and pair $xy \in 2_2^N$ such that $xy \notin I_i$, there exists some admissible preference function $R_i \in \mathcal{R}(I_i)$ with $R_i(x) = R_i(y)$.*

The intuition for this result is straightforward and we omit the proof. Next, we propose an example.

Example 3 *Suppose that $N = \{a, b, c\}$. Let I be a jury configuration such that, for some expert i, $ac \notin I_i$. In particular, suppose that $I_i = \{ab, bc\}$. Table 3 shows an example admissible preference function for expert i, $R_i \in \mathcal{R}(I_i)$, where $R_i(a) = R_i(c)$.*

R_i

a	**b**	**c**
ac	b	ac
b	ac	b

Table 3 Preference function R_i in Example 3.

Our second lemma states that, if the preference relations of each expert are the same in two different states, then every SCF that is implementable in some equilibrium concept must select the same contestant in both states.

Lemma 2 *Let $F \in \mathcal{F}(\delta)$ be a SCF implementable in $\mathcal{E}$-equilibrium for some $\mathcal{E} \in \mathbb{E}$. Let $(R, w), (\hat{R}, \hat{w}) \in \mathcal{R}(I) \times W(\delta)$ be two admissible states such that $R_i(w_i) = \hat{R}_i(\hat{w}_i)$ for every $i \in E$. Then, $F(w) = F(\hat{w})$.*

Proof. Suppose that F is implementable in $\mathcal{E}$-equilibrium using a mechanism $\Gamma = (M, g)$. Suppose that (R, w) and $(\hat{R}, \hat{w})$ are such that the preference relations of each expert i at both states are the same, *i.e.*, $R_i(w_i) = \hat{R}_i(\hat{w}_i)$ for every $i \in E$. Then, a profile of messages $m \in M$ is an $\mathcal{E}$-equilibrium of Γ at state (R, w) if and only if it is an $\mathcal{E}$-equilibrium of Γ at state $(\hat{R}, \hat{w})$, *i.e.*, $\mathcal{E}(\Gamma, R, w) = \mathcal{E}(\Gamma, \hat{R}, \hat{w})$. Suppose by contradiction that $F(w) \neq F(\hat{w})$. Since Γ implements F in $\mathcal{E}$-equilibrium, there exists $m \in \mathcal{E}(\Gamma, R, w)$ such that $g(m) = F(w)$. Then $m \in \mathcal{E}(\Gamma, \hat{R}, \hat{w})$ and $g(m) \neq F(\hat{w})$, which contradicts that Γ implements F in $\mathcal{E}$-equilibrium. ■

Next, we define the crucial concept of an expert being decisive. We say that an expert i is decisive in a SCF F at a pair of contestants xy if, for some given fixed opinions of the rest of experts, the contestants selected by F for the cases $w_i = x$ and $w_i = y$ are different.

Definition 7 *Suppose that $\delta > 1$. An expert $i \in E$ is* ***decisive*** *in $F \in \mathcal{F}(\delta)$ at $xy \in 2^N_2$ if there exist $w, \hat{w} \in W(\delta)$ such that:*
(i) $w_i = x$,
(ii) $\hat{w}_i = y$,
(iii) $w_j = \hat{w}_j$ for every $j \in E \backslash \{i\}$, and
(iv) $F(w) \neq F(\hat{w})$.

If F is implementable in some equilibrium concept, then each expert must be impartial with respect to each pair of contestants in which she is decisive. The idea is that, for F to be implementable, the preference relation of at least one expert must change between any two states (R, w) and $(R, \hat{w})$ such that $F(w) \neq F(\hat{w})$. If w and $\hat{w}$ only differ in that $w_i = x$ and $\hat{w}_i = y$, then the only possibility is that i is impartial with respect to xy.

Lemma 3 *Let $F \in \mathcal{F}(\delta)$ be a SCF implementable in $\mathcal{E}$-equilibrium for some $\mathcal{E} \in \mathbb{E}$. If an expert $i \in E$ is decisive in F at some pair of contestants $xy \in 2^N_2$, then $xy \in I_i$.*

Proof. Let $F \in \mathcal{F}(\delta)$ be a SCF implementable in $\mathcal{E}$-equilibrium. By contradiction, suppose that there exists $i \in E$, $xy \in 2^N_2$, and $w, \hat{w} \in W(\delta)$

such that (i) $w_i = x$, (ii) $\hat{w}_i = y$, (iii) $w_j = \hat{w}_j$ for every $j \in E\backslash\{i\}$, (iv) $F(w) \neq F(\hat{w})$, and (v) $xy \notin I_i$. Let $R \in \mathcal{R}(I)$ be such that $R_i(x) = R_i(y)$ (by Lemma 1, such a preference function exists because $xy \notin I_i$). Note that $(R, w), (R, \hat{w}) \in \mathcal{R}(I) \times W(\delta)$ are such that $R_j(\hat{w}_j) = R_j(w_j)$ for every $j \in E$ (i included). Then, because F is implementable in $\mathcal{E}$-equilibrium, by Lemma 2 we have $F(w) = F(\hat{w})$, which is a contradiction. ■

Now we can state our main results. First, we show that if the number of experts is odd and different experts may have different opinions about who is the best contestant, then no majoritarian SCF can be implemented, regardless of the equilibrium concept considered, unless the jury is totally impartial. The intuition of this result is simple. Given any expert i and any pair of contestants xy, let w be such that a minimum majority of $\left\lceil \frac{|E|}{2} \right\rceil$ experts, with i among them, believe that the best contestant is x, while the other $\left\lfloor \frac{|E|}{2} \right\rfloor$ experts believe that the best contestant is y.[5] Let $\hat{w}$ be equal to w except that $\hat{w}_i = y$, so that now a minimum majority of $\left\lceil \frac{|E|}{2} \right\rceil$ experts believe that the best contestant is y. If F is majoritarian then $F(w) = x$ and $F(\hat{w}) = y$, and therefore i is decisive at xy. Then, by Lemma 3, if F is implementable in some equilibrium concept i must be impartial with respect to xy.

Theorem 1 *Suppose that $\delta > 1$ and $|E|$ is odd. Let $F \in \mathcal{F}^M(\delta)$ be a majoritarian SCF. If F is implementable in $\mathcal{E}$-equilibrium for some $\mathcal{E} \in \mathbb{E}$, then the jury configuration is totally impartial.*

Proof. Let $\delta > 1$ and let $|E|$ be odd. Given a jury configuration I, let $F \in \mathcal{F}^M(\delta)$ be implementable in $\mathcal{E}$-equilibrium for some $\mathcal{E} \in \mathbb{E}$. By contradiction, suppose that there exists $xy \in 2^N_2$ and $i \in E$ such that $xy \notin I_i$. Let $w \in W(\delta)$ be such that (i) $w_i = x$, (ii) $|E^x_w| = \left\lceil \frac{|E|}{2} \right\rceil$, and (iii) $|E^y_w| = \left\lfloor \frac{|E|}{2} \right\rfloor$ (*i.e.*, $\left\lceil \frac{|E|}{2} \right\rceil$ experts, including i, think that x is the best contestant, while $\left\lfloor \frac{|E|}{2} \right\rfloor$ experts think that y is the best contestant). Because $|E|$ is an odd number, then $|E^x_w| > \frac{|E|}{2}$. Then, because F is majoritarian, $F(w) = x$. Let $\hat{w} \in W(\delta)$ be such that (i) $\hat{w}_i = y$ and (ii) $\hat{w}_j = w_j$ for every $j \in E\backslash\{i\}$. Note that

[5] For each real number $\alpha \in \mathbb{R}$, $\lceil \alpha \rceil = \min\{\beta \in \mathbb{Z} : \beta \geq \alpha\}$ and $\lfloor \alpha \rfloor = \max\{\beta \in \mathbb{Z} : \beta \leq \alpha\}$, where $\mathbb{Z}$ is the set of integers.

$|E^y_{\hat{w}}| > \frac{|E|}{2}$ and, since F is majoritarian, then $F(\hat{w}) = y$. Then, expert i is decisive in F at pair xy. By Lemma 3, because F is implementable in $\mathcal{E}$-equilibrium, we have $xy \in I_i$, which is a contradiction. ■

In many economic problems it is unrealistic to believe that the jury configuration is totally impartial as, for example, some of the experts have friends or enemies among the contestants. Therefore, Theorem 1 can be interpreted as showing that, if the number of experts is odd and different experts may have different opinions, it is not possible to implement any majoritarian SCF in any concept of equilibrium. Note that, for this impossibility result to hold, the opinions of the experts do not need to be "very different" since it suffices that $\delta = 2$. Next, we show an example of this result for the three experts and three contestants case.

Example 4 *Suppose that $E = \{1, 2, 3\}$, $N = \{a, b, c\}$, and $\delta = 2$. Let I be a jury configuration be such that $I_1 = \{ab, bc, ac\}$, $I_2 = \{bc, ac\}$, and $I_3 = \{ab, bc, ac\}$. Note that I is not totally impartial because $ab \notin I_2$. Let $F \in \mathcal{F}^M(\delta)$, $w = aab$, and $\hat{w} = abb$. Because F is majoritarian then $a = F(w) \neq F(\hat{w}) = b$. Therefore, expert 2 is decisive in F at pair ab. Since $ab \notin I_2$, Lemma 3 implies that F is not $\mathcal{E}$-implementable in any equilibrium concept $\mathcal{E} \in \mathbb{E}$. To see this, consider, for example, a profile of admissible preference functions $R \in \mathcal{R}(I)$ where R_2 is as described in Table 4. Note that $R_i(w_i) = R_i(\hat{w}_i)$ for every $i \in E$. Then, $\mathcal{E}(\Gamma, R, w) = \mathcal{E}(\Gamma, R, \hat{w})$. Since Γ implements F in $\mathcal{E}$-equilibrium, there exists $m \in \mathcal{E}(\Gamma, R, w)$ such that $g(m) = F(R, w) = a$. But then $m \in \mathcal{E}(\Gamma, R, \hat{w})$ and $g(m) \neq F(R, \hat{w}) = b$, which contradicts that Γ implements F in $\mathcal{E}$-equilibrium.*

R_2

a	**b**	**c**
a	a	c
b	b	ba
c	c	

Table 4 Preference function R_2 in Example 4.

The argument used to prove Theorem 1 cannot be used when the number of experts is even. The reason is that, in this case, if some contestant x is viewed as the best contestant by a majority of experts, the change of opinion of a single expert cannot make a different contestant y be viewed as the best contestant by a new majority (at best, the change of opinion of a single expert could cause a tie between x and y). This is the reason why, in the result analogous to Theorem 1 for the case where $|E|$ is even, total impartiality is replaced by quasi-impartiality. Our next theorem shows that, if $|E|$ is even and $\delta > 1$, then no majoritarian SCF can be implemented, regardless of the equilibrium concept considered, unless the jury configuration is quasi-impartial. The idea behind this result is the following. Suppose that F is majoritarian SCF that is implementable in some equilibrium concept and i is an expert who is not impartial which respect to some pair of contestants xy. It turns out that, if w is such that i and other $\frac{|E|}{2}-1$ experts believe that the best contestant is x and the other $\frac{|E|}{2}$ experts believe that the best contestant is y, then $F(w)=y$. Suppose by contradiction that there is another expert j who is also not impartial with respect to xy. Then, if j is among the experts who believe that the best contestant is y in w, we have $F(w)=x$, which is a contradiction. Although quasi-impartiality is weaker than total impartiality, it is still a very demanding requirement.

Theorem 2 *Suppose that $\delta > 1$ and $|E|$ is even. Let $F \in \mathcal{F}^M(\delta)$ be a majoritarian SCF. If F is implementable in $\mathcal{E}$-equilibrium for some $\mathcal{E} \in \mathbb{E}$, then the jury configuration is quasi-impartial.*

Proof. Let $\delta > 1$ and let $|E|$ be even. Given a jury configuration I, let $F \in \mathcal{F}^M(\delta)$ be implementable in $\mathcal{E}$-equilibrium for some $\mathcal{E} \in \mathbb{E}$.

Step 1. For every $xy \in 2^N_2$, $i \in E$, and $w \in W(\delta)$ such that (i) $xy \notin I_i$, (ii) $w_i = x$, and (iii) $|E^x_w| = |E^y_w| = \frac{|E|}{2}$, we have $F(w)=y$.

Suppose by contradiction that $F(w) \neq y$. Let $\hat{w} \in W(\delta)$ be such that (i) $\hat{w}_i = y$, (ii) $\hat{w}_j = w_j$ for every $j \in E\backslash\{i\}$. Then, $|E^y_{\hat{w}}| > \frac{|E|}{2}$ and, because F is majoritarian, $F(\hat{w}) = y$. Hence, expert i is decisive in F at pair xy. By Lemma 3, because F is implementable in $\mathcal{E}$-equilibrium, we have $xy \in I_i$, which is a contradiction.

Step 2. If $xy \in 2^N_2$ and $i \in E$ are such that $xy \notin I_i$, then $xy \in I_j$ for every $j \in N\backslash\{i\}$.

Suppose that there exists $xy \in 2^N_2$ and $i, j \in E$ such that $xy \notin I_i$ and $xy \notin I_j$. Let $w \in W(\delta)$ be such that (i) $w_i = x$, (ii) $w_j = y$, and (iii) $|E^x_w| = |E^y_w| = \frac{|E|}{2}$. Because $xy \notin I_i$ and $w_i = x$, by Step 1 we have $F(w)=y$. Similarly, because $xy \notin I_j$ and $w_j = y$, by Step 1 we have $F(w)=x$, which is a contradiction. ■

The results stated in Theorems 1 and 2 are very consistent and still hold if we replace majoritarianism by two other reasonable properties: respect for the jury and anonymity. Respect for the jury is related to majoritarianism and requires that the contestant selected by the SCF must be considered as the best contestant by at least one expert.

Definition 8 *A SCF $F \in \mathcal{F}(\delta)$ satisfies* ***respect for the jury*** *if, for every $w \in W(\delta)$, $\left|E_w^{F(w)}\right| > 0$. Let $\mathcal{F}^R(\delta) \subset \mathcal{F}(\delta)$ denote the set of all SCFs satisfying this property.*[6]

Anonymity requires that changing the names of the experts with each opinion would not change the contestant considered to be the deserving winner. A permutation of a set is a one-to-one function of that set into itself. For any admissible profile of experts' opinions $w \in W(\delta)$ and any permutation $\pi : E \to E$ of the set of experts, let $w \bullet \pi$ be the jury observation derived from w by assigning to each expert i the observation of expert $\pi(i)$ in w, that is, $(w \bullet \pi)_i = w_{\pi(i)}$.

Definition 9 *A SCF $F \in \mathcal{F}(\delta)$ is* ***anonymous*** *if, for every permutation $\pi : E \to E$ and every profile of experts' opinions $w \in W(\delta)$, $F(w \bullet \pi) = F(w)$. Let $\mathcal{F}^A(\delta) \subset \mathcal{F}(\delta)$ denote the set of all anonymous SCFs.*

Next we show that, regardless of whether $|E|$ is odd or even, total impartiality is a necessary condition for implementation if we replace majoritarianism by respect for the jury and anonymity. The intuition of this result is as follows. For each possible pair of contestants xy, consider a sequence of profiles of experts' opinions $\{w^t\}_{t=0}^{|E|}$ where each w^t is such that each expert $i > t$ believes that the best contestant is x and each expert $i \leq t$ believes that the best contestant is y. By respect for the jury, $F(w^0) = x$, $F(w^{|E|}) = y$, and $F(w^t) \in \{x, y\}$ for every t. Therefore, there is one element of the sequence, w^{j^*}, with $F(w^{j^*-1}) = x$ and $F(w^{j^*}) = y$, which implies that expert j^* is decisive at xy. Then, if F is implementable in some equilibrium concept, j^* is impartial with respect to xy. By anonymity, we can make j^* be any expert i.

[6]If $\delta = 1$, majoritarianism and respect for the jury are equivalent. If $|E|$ is odd and $\delta = 2$, then $\mathcal{F}^M(\delta) \subset \mathcal{F}^R(\delta)$. However, in general, neither majoritarianism implies respect for the jury, nor respect for the jury implies majoritarianism.

Theorem 3 *Suppose that $\delta > 1$. Let $F \in \mathcal{F}^R(\delta) \cap \mathcal{F}^A(\delta)$ be a SCF satisfying respect for the jury and anonymity. If F is implementable in $\mathcal{E}$-equilibrium for some $\mathcal{E} \in \mathbb{E}$, then the jury configuration is totally impartial.*

Proof. Let $\delta > 1$. Given any jury configuration I, let $F \in \mathcal{F}^R(\delta) \cap \mathcal{F}^A(\delta)$ be implementable in $\mathcal{E}$-equilibrium for some $\mathcal{E} \in \mathbb{E}$. By contradiction, suppose that there exists $xy \in 2_2^N$ and $i \in E$ such that $xy \notin I_i$. Let $w^0 \in W(\delta)$ be such that $w_j^0 = x$ for every $j \in E$. Because F satisfies respect for the jury, $F(w^0) = x$. Let $w^1 \in W(\delta)$ be such that $w_1^1 = y$ and $w_j^1 = x$ for every $j \in E \backslash \{1\}$. Because F satisfies respect for the jury, either $F(w^1) = x$ or $F(w^1) = y$. Suppose that $F(w^1) = x$. Let $w^2 \in W(\delta)$ be such that $w_1^2 = w_2^2 = y$ and $w_j^2 = x$ for every $j \in E \backslash \{1, 2\}$. Again, because F satisfies respect for the jury, either $F(w^2) = x$ or $F(w^2) = y$. Continuing with this process, we have a sequence $(w^j)_{j=0}^{|E|}$, where each $w^j \in W(\delta)$ is such that, for all $k \in E$, (i) $w_k^j = y$ if $k \leq j$, and (ii) $w_k^j = x$ if $k > j$. Note that the last element of this sequence, $w^{|E|}$, is such that $w_k^{|E|} = y$ for every $k \in E$, and then, because F satisfies respect for the jury, $F(w^{|E|}) = y$. Thus, we can conclude that there is one element of the sequence, w^{j^*} with $j^* \neq 0$, such that $F(w^{j^*-1}) = x$ and $F(w^{j^*}) = y$. Note that $w_{j^*}^{j^*-1} = x$, $w_{j^*}^{j^*} = y$, $w_k^{j^*-1} = w_k^{j^*}$ for every $k \in E \backslash \{j^*\}$, and $F(w^{j^*-1}) \neq F(w^{j^*})$. Hence, expert j^* is decisive in F at pair xy. Consider now a permutation $\pi : E \to E$ of the set of experts such that $\pi(j^*) = i$, $\pi(i) = j^*$, and $\pi(k) = k$ for every $k \in E \backslash \{j^*, i\}$. Because F satisfies anonymity, $F(w^{j^*-1} \bullet \pi) = F(w^{j^*-1}) = x$ and $F(w^{j^*} \bullet \pi) = F(w^{j^*}) = y$. Note that $(w^{j^*-1} \bullet \pi)_i = x$, $(w^{j^*} \bullet \pi)_i = y$, $(w^{j^*-1} \bullet \pi)_k = (w^{j^*} \bullet \pi)_k$ for every $k \in E \backslash \{i\}$, and $F(w^{j^*-1} \bullet \pi) \neq F(w^{j^*} \bullet \pi)$. Therefore, expert i is decisive in F at xy. Then, by Lemma 3, because F is implementable in $\mathcal{E}$-equilibrium, we have $xy \in I_i$, which is a contradiction. ■

4 A particular case: implementation in Nash equilibrium

In the previous section we have established necessary conditions on the jury configuration for the existence of a majoritarian SCF that is implementable in some equilibrium concept. Whether these conditions are also sufficient or not may depend on the equilibrium concept considered. In this section we study the particular case of Nash implementation. We begin by establishing a necessary and sufficient condition for a majoritarian SCF to be implementable in Nash equilibrium: if the contestant who is considered deserving winner at some profile of experts' opinions w is not considered deserving winner at some other profile $\hat{w}$, then there must be at least one expert i who (1) agrees that is $F(w)$ is the best contestant at w (*i.e.*, $w_i = F(w)$), (2) has a different opinion about who is the best contestant at $\hat{w}$ (*i.e.*, $\hat{w}_i \neq w_i$), and (3) is impartial with respect these two contestants (*i.e.*, $w_i\hat{w}_i \in I_i$). In our setting, this condition is equivalent to the well-known condition of Maskin monotonicity (Maskin, 1999). Maskin monotonicity together with a no veto power condition (which is trivially satisfied by any majoritarian SCF) are sufficient conditions for Nash implementation when there are at least three agents.

Lemma 4 *Let $F \in \mathcal{F}(\delta)$ be a SCF implementable in Nash equilibrium. Then, for every $w, \hat{w} \in W(\delta)$ with $F(w) \neq F(\hat{w})$, there exist $i \in E$ such that $F(w) = w_i$ and $w_i\hat{w}_i \in I_i$. Moreover, if F is majoritarian and there are at least three experts, the previous condition is also sufficient for the Nash implementability of F.*

Proof. First we prove that, given any jury configuration I, a SCF $F \in \mathcal{F}(\delta)$ satisfies the condition of the statement if and only if it satisfies Maskin monotonicity, a necessary condition for implementation in Nash equilibrium (Maskin, 1999). Maskin monotonicity requires that, for every two states $(R, w), (\hat{R}, \hat{w}) \in \mathcal{R}(I) \times W(\delta)$, if $F(w) \neq F(\hat{w})$, then there exist $i \in E$ and $x \in N$ such that $F(w)\ R_i(w_i)\ x$ and $x\ \hat{P}_i(\hat{w}_i)\ F(w)$. Let $(R, w), (\hat{R}, \hat{w}) \in \mathcal{R}(I) \times W(\delta)$ be such that $F(w) \neq F(\hat{w})$. From the condition of the statement, there is $i \in E$ be such that $F(w) = w_i$ and $w_i\hat{w}_i \in I_i$. Because $w_i\hat{w}_i \in I_i$ and $R_i, \hat{R}_i \in \mathcal{R}(I_i)$, $w_i\ R_i(w_i)\ \hat{w}_i$ and $\hat{w}_i\ \hat{P}_i(\hat{w}_i)\ w_i$. Therefore, $F(w)\ R_i(w_i)\ \hat{w}_i$ and $\hat{w}_i\ \hat{P}_i(\hat{w}_i)\ F(w)$, which implies that Maskin monotonicity is satisfied. Now, we prove that if F satisfies Maskin monotonicity, the it satisfies the condition of the statement. Suppose by contradiction that F satisfies Maskin monotonicity but there are $w, \hat{w} \in W(\delta)$ with $F(w) \neq F(\hat{w})$ such that, for every $i \in E$ with $F(w) = w_i$, we have $w_i\hat{w}_i \notin I_i$. Note that, for every $i \in E$ with either (i) $F(w) \neq w_i$ or (ii) $F(w) = w_i$ and $w_i\hat{w}_i \notin I_i$, there exists $R_i \in \mathcal{R}(I_i)$ such that, for every $x \in N$, if $F(w)\ R_i(w_i)$ x then $F(w)\ R_i(\hat{w}_i)\ x$. Therefore, there exists some $R \in \mathcal{R}(I)$ such that, for every $i \in E$ and every $x \in N$, if $F(w)\ R_i(w_i)\ x$ then $F(w)\ R_i(\hat{w}_i)\ x$. Hence, $(R, w), (R, \hat{w}) \in \mathcal{R}(I) \times W(\delta)$ are such that $F(w) \neq F(\hat{w})$ and, for every $i \in E$ and every $x \in N$, if $F(w)\ R_i(w_i)\ x$ then $F(w)\ R_i(\hat{w}_i)\ x$. This contradicts that F satisfies Maskin monotonicity. With at least three experts, Maskin monotonicity plus a condition called no veto power is sufficient for implementation in Nash equilibrium (Maskin 1999). No veto power requires that, for every $(R, w) \in \mathcal{R}(I) \times W(\delta)$, $x \in N$, and $j \in E$, if $x\ R_j(w_j)\ y$ for every $y \in N$ and every $i \in E \backslash \{j\}$ then $x = F(w)$. Note that if $F \in \mathcal{F}^M(\delta)$ then it satisfies no veto power. ■

Next we show that, when implementing a majoritarian SCF in Nash equilibrium, there is a big difference between the cases where the limit of opinions is two or greater than two.

4.1 When the limit of opinions is $\delta = 2$

Suppose we know that any possible discrepancy between the experts about who is the best contestant will be limited to a maximum of two contestants (i.e., we know that there are at most two different opinions in the jury). We begin by analyzing the case in which $|E|$ is an odd number. In this case, having a totally impartial jury configuration is not just a necessary condition for the existence of a majoritarian SCF that is implementable in some equilibrium concept (as shown in Theorem 1), but it is also a sufficient condition when the equilibrium concept considered is Nash equilibrium and $|E| \geq 3$. In fact, under these requirements, every majoritarian SCF is Nash implementable, as it satisfies the sufficient condition stated in Lemma 4.

Theorem 4 *Suppose that $\delta = 2$. Suppose that $|E| \geq 3$ is an odd number. A majoritarian SCF is implementable in Nash equilibrium if and only if the jury configuration is totally impartial.*

The proof of Theorem 4 is given in the Appendix. Now we analyze the case where $|E|$ is even. From Theorem 2 we know that quasi-impartiality is a necessary condition for the implementability of any majoritarian SCF when $\delta = 2$ and $|E|$ is even. However, this condition is not sufficient when the equilibrium concept considered is Nash equilibrium. In order to guarantee the existence of a majoritarian SCF that is implementable in Nash equilibrium, the jury configuration must satisfy a stronger condition that we call strict-quasi-impartiality. This new condition requires at least $|E| - 1$ experts to be impartial with respect to all pair of contestants. Given a jury configuration

I, let $E^* = \{i \in E : xy \in I_i \text{ for every } xy \in 2_2^N\}$ be the set of experts who are impartial with respect to every pair of contestants.

Definition 10 *A jury configuration is* ***strictly-quasi-impartial*** *if* $|E^*| \geq |E| - 1$.[7]

Our next theorem shows that strict-quasi-impartiality is a necessary and sufficient condition for a majoritarian and Nash implementable SCF to exist when $\delta = 2$ and $|E|$ is even.

Theorem 5 *Suppose that* $\delta = 2$*. Suppose that* $|E| \geq 3$ *is an even number. A majoritarian and Nash implementable SCF exists if and only if the jury configuration is strictly-quasi-impartial.*

The formal proof of Theorem 5 is given in the Appendix. The idea behind the necessity part is that, if I is not strictly-quasi-impartial, we can find two profiles of experts' opinions, w and $\hat{w}$, such that $F(w) \neq F(\hat{w})$ and, for all expert i, $w_i \neq F(w)$ (and therefore, the necessary condition for Nash implementation stated in Lemma 4 is not satisfied). To prove the sufficient part, we propose a majoritarian SCF that is implementable in Nash equilibrium if the jury configuration is strictly-quasi-impartial. This SCF selects the contestant who is viewed as the best one by a majority of experts whenever it exists. If there is a tie and I is totally impartial, the SCF selects the contestant who is viewed as the best one by expert 1. If there is a tie and I is not totally impartial, the SCF selects, from among the contestants involved in the tie, the one who is not viewed as the best contestant by the only expert who is not impartial with respect to some pair of contestants.

Unlike what happens when $|E|$ is odd, the necessary and sufficient condition for the existence of a majoritarian and Nash implementable SCF when $|E|$ is even does not guarantee that every majoritarian SCF is Nash implementable.

Remark 1 *Suppose that* $\delta = 2$, $|E| \geq 3$ *is an even number, and the jury configuration is strictly-quasi-impartial. The fact that a SCF is majoritarian does not necessarily imply that it is implementable in Nash equilibrium. We proof this remark in the Appendix.*

[7] Strict-quasi-impartiality is a condition stronger than quasi-impartiality and weaker than total impartiality.

4.2 When the limit of opinions is $\delta > 2$

If the limit of opinions is greater than two, then no majoritarian SCF can be implemented in Nash equilibrium, regardless of the properties that the jury configuration may satisfy.

Theorem 6 *If $\delta > 2$, there is no majoritarian SCF implementable in Nash equilibrium for any jury configuration.*

The proof of Theorem 6 is in the Appendix. The intuition is that, if $\delta > 2$ then, for every majoritarian SCF (and regardless of how the jury configuration is), we can find always two different profiles of experts' opinions w and $\hat{w}$ with $F(w) \neq F(\hat{w})$ and such that, for every i with $w_i = F(w)$, $\hat{w}_i = w_i$. Therefore, the necessary condition for Nash implementation stated in Lemma 4 is not satisfied.

5 Conclusion

We have studied the problem of aggregating the honest opinions of a group of experts to determine who is the deserving winner of a competition. The aggregation procedure is said to be majoritarian if, whenever more than fifty percent of the experts honestly believe that certain contestant is the best one, then that contestant is socially considered the deserving winner. The fact that an expert believes that a contestant is the best one does not necessarily imply that she wants this contestant to win as, for example, she may be biased in favor of some other contestant. A biased expert might not want to reveal her honest opinion about who is the best contestant. For this reason, we have to design a mechanism (*e.g.*, a voting system) that gives the incentives to the experts to always choose the contestant who is socially considered deserving winner according to their honest opinions. Unfortunately, if the process of aggregation of experts' honest opinions is majoritarian, the necessary conditions for the existence of such mechanism are very strong. Although there are small differences depending on whether the number of experts is even or odd, the basic requirement is that the group of experts must be totally impartial. This can be interpreted as an impossibility result in many economic problems as, for example, it eliminates the possibility that experts have friends who want to favor among the contestants. The result is very consistent since (1) it does not depend on the equilibrium concept considered, (2) for the result to hold it is sufficient that there are two different opinions among the experts, and (3) the result still holds if we replace majoritarianism by two other reasonable properties: respect for the jury (the deserving winner must be considered as the best contestant by at least one expert) and anonymity (changing the names of the experts with each opinion would not change the deserving winner). As strong as it is, total impartiality is only a necessary condition for implementing the deserving winner when the process of aggregation of experts' honest opinions is majoritarian. Whether this condition is sufficient or not may depend on the equilibrium concept considered. We have studied Nash equilibrium as a particular case and showed that, if we allow for the possibility that more than two experts have different opinions, then the impossibility result is still stronger: no majoritarian process of aggregation of experts' opinions can be implemented with any mechanism, even if the group of experts is totally impartial.

Appendix

PROOF OF THEOREM 4. Suppose $\delta = 2$ and $|E|$ is odd. Given any jury configuration I, from Theorem 1 we know that if $F \in \mathcal{F}^M(\delta)$ is implementable in Nash equilibrium then I is totally impartial. Next we prove that if $|E| \geq 3$ is odd, $\delta = 2$, and I is totally impartial, then every SCF $F \in \mathcal{F}^M(\delta)$ satisfies the necessary and sufficient condition for Nash implementation stated in Lemma 4. To see this note that, because $\delta = 2$, $|E| \geq 3$ is an odd number, and $F \in \mathcal{F}^M(\delta)$, then at least $\left\lceil \frac{|E|}{2} \right\rceil$ experts observe $F(w)$ as the best contestant in w. Because $\left\lceil \frac{|E|}{2} \right\rceil > \frac{|E|}{2}$, if $F(w) \neq F(\hat{w})$, then there is at least one expert i such that $w_i = F(w)$ and $\hat{w}_i \neq F(w)$. Moreover, because I is totally impartial, $w_i\hat{w}_i \in I_i$. ■

PROOF OF THEOREM 5. Suppose that $\delta = 2$ and $|E|$ is even.

Claim 1. If $F \in \mathcal{F}^M(\delta)$ is implementable in Nash equilibrium, then I is strictly-quasi-impartial.

Step 1. For every $i \in E$, $xy \in 2^N_2$ such that $xy \notin I_i$, and $w \in W(\delta)$ such that (i) $w_i = x$, and (ii) $|E^x_w| = |E^y_w| = \frac{|E|}{2}$, we have $F(w) = y$.

The proof of this statement follows from Step 1 in the proof of Theorem 2.

Step 2. For every $i \in E$, $xy \in 2^N_2$ such that $xy \notin I_i$, $z \in N$, and $w \in W(\delta)$ such that (i) $w_i = x$, and (ii) $|E^x_w| = |E^z_w| = \frac{|E|}{2}$, we have $F(w) \neq x$.

Suppose, on the contrary, that $F(w) = x$. Let $\hat{w} \in W(\delta)$ be such that (i) $\hat{w}_i = y$, (ii) $\hat{w}_j = y$ for every $j \in E$ with $w_j = z$, and (iii) $\hat{w}_j = x$ for every $j \in E\backslash\{i\}$ with $w_j = x$. Because $|E^y_{\hat{w}}| > \frac{|E|}{2}$ and $F \in \mathcal{F}^M(\delta)$, $F(\hat{w}) = y$. Then, $F(w) \neq F(\hat{w})$ and there is no $j \in E$ such that $F(w) = w_j$ and $w_j\hat{w}_j \in I_j$. By Lemma 4, this contradicts that F is implementable in Nash equilibrium.

Step 3. For every $i \in E$, $xy \in 2^N_2$ such that $xy \notin I_i$, $j \in E\backslash\{i\}$, and $z \in N\backslash\{x\}$, we have $xz \in I_j$.

Suppose, on the contrary, that there exist $i \in E$, $xy \in 2^N_2$ with $xy \notin I_i$, $j \in E\backslash\{i\}$, and $z \in N$ such that $xz \notin I_j$. From Theorem 2 I is quasi-impartial and then $z \neq y$. Let $w \in W(\delta)$ be such that $w_i = y$, $w_j = x$, and $|E^x_w| = |E^y_w| = \frac{|E|}{2}$. Because $xy \notin I_i$, $w_i = y$, and $|E^x_w| = |E^y_w| = \frac{|E|}{2}$ then, by Step 1, $F(w) = x$. However, because $xz \notin I_j$, $w_j = x$, and $|E^x_w| = |E^y_w| = \frac{|E|}{2}$ then, by Step 2, $F(w) \neq x$, which is a contradiction.

Step 4. For every $i \in E$, $xy \in 2^N_2$ such that $xy \notin I_i$, $j \in E\backslash\{i\}$, and $\hat{x}\hat{y} \in 2^N_2$ we have $\hat{x}\hat{y} \in I_j$.

Suppose on the contrary that there exist $i \in E$, $xy \in 2^N_2$ with $xy \notin I_i$, $j \in E\backslash\{i\}$, and $\hat{x}\hat{y} \in 2^N_2$ such that $\hat{x}\hat{y} \notin I_j$. From Step 3 we have $x\hat{y} \in I_j$ for every $\hat{y} \in N$ and $\hat{x}y \in I_j$ for every $\hat{x} \in N$. Therefore, $\hat{x} \notin \{x, y\}$ and $\hat{y} \notin \{x, y\}$. Let $w \in W(\delta)$ be such that (i) $w_i = x$, (ii) $w_j = \hat{x}$, and (iii) $|E^x_w| = \left|E^{\hat{x}}_w\right| = \frac{|E|}{2}$. Because $xy \notin I_i$ and $w_i = x$, by Step 2, $F(w) \neq x$. Because $\hat{x}\hat{y} \notin I_j$ and $w_j = \hat{x}$, by Step 2, $F(w) \neq \hat{x}$. Therefore, $F(w) = z$ for some $z \notin \{x, \hat{x}\}$. Given any $\hat{z} \in N\backslash\{z\}$, let $\hat{w} \in W(\delta)$ be such that $\hat{w}_k = \hat{z}$ for every $k \in E$. Because $F \in \mathcal{F}^M(\delta)$ then $F(\hat{w}) = \hat{z}$. Then, $F(w) \neq F(\hat{w})$ and there is no $k \in E$ such that $F(w) = w_k$ and $w_k\hat{w}_k \in I_k$. By Lemma 4, this contradicts that F is implementable in Nash equilibrium.

From Step 4 we have that, if $xy \notin I_i$ for some $xy \in 2^N_2$ and some $i \in E$, then $\hat{x}\hat{y} \in I_j$ for every $\hat{x}\hat{y} \in 2^N_2$ and every $j \in E\backslash\{j\}$, and therefore I is strictly-quasi-impartial.

Claim 2. If $|E| \geq 3$ and I is strictly-quasi-impartial, then a SCF $F^* \in \mathcal{F}^M(\delta)$ exists that is implementable in Nash equilibrium.

Because I is strictly-quasi-impartial, there is at most one expert i^* such that $xy \notin I_{i^*}$ for some $xy \in 2^N_2$. Let $F^* \in \mathcal{F}(2)$ be such that, for each $w \in W(2)$, (1) if $|E^x_w| > \frac{|E|}{2}$ for some some $x \in N$ then $F^*(w) = x$, and (2) if $|E^x_w| = |E^y_w| = \frac{|E|}{2}$ for some $x, y \in N$ then, (2.1) if I is totally impartial then then $F^*(w) = w_1$, and (2.2) if I is not totally impartial, then $w_{i^*} \neq F^*(w) \in \{x, y\}$. Clearly, F^* is majoritarian.[8] Next we show that F^* satisfies the necessary and sufficient condition for Nash implementation stated in Lemma 4. Let $w, \hat{w} \in W(2)$ be such that $F^*(w) \neq F^*(\hat{w})$. Suppose first that $|E^x_w| > \frac{|E|}{2}$ for some $x \in N$. In this case, by definition of F^* and because $F^*(w) \neq F^*(\hat{w})$, there is at least one expert $i \neq i^*$ such that $w_i = x = F^*(w)$ and $\hat{w}_i \neq x$. Because all experts different from i^* are impartial with respect to every pair of contestants, then $w_i\hat{w}_i \in I_i$. Suppose now that $|E^x_w| = |E^y_w| = \frac{|E|}{2}$ for some $x, y \in N$ and I. If I is totally impartial then $F^*(w) = w_1$. Suppose without loss of generality that $w_1 = x$. By definition of F^* and because $F^*(w) = x \neq F^*(\hat{w})$, there is at least one expert i such that $w_i = x$ and $\hat{w}_i \neq x$. Moreover, because I is totally impartial, then $w_i\hat{w}_i \in I_i$. If I is not totally impartial, then $w_{i^*} \neq F^*(w) \in \{x, y\}$. Suppose without loss of generality that $w_{i^*} = y$. Then $F^*(w) = x$. Again, by definition of F^* and because $F^*(w) = x \neq F^*(\hat{w})$, there is at least one expert i such that $w_i = x$

[8] It satisfies also respect for the jury, but it is not anonimous.

and $\hat{w}_i \neq x$. Because $w_{i^*} = y$, then $i \neq i^*$, and because all experts different from i^* are impartial with respect to all possible pair of contestants, then $w_i\hat{w}_i \in I_i$. ■

PROOF OF REMARK 1. Let $i^* \in E$ be the only expert such that $xy \notin I_{i^*}$ for some $xy \in 2_2^N$. Let $\tilde{F} \in \mathcal{F}(2)$ be a SCF such that, for each $w \in W(2)$, (1) if $|E_w^x| > \frac{|E|}{2}$ for some some $x \in N$ then $\tilde{F}(w) = x$, and (2) if $|E_w^x| = |E_w^y| = \frac{|E|}{2}$ for some $x, y \in N$ then $\tilde{F}(w) = w_{i^*}$ (*i.e.*, $\tilde{F}$ selects the contestant who is viewed as the best one by a majority of experts whenever it exists; if there is a tie, $\tilde{F}$ selects the contestant viewed as the best one by the only expert who is not impartial with respect to some pair of contestants). Note that $\tilde{F}$ is majoritarian. Let $xy \notin I_{i^*}$. Let $w \in W(2)$ be such that $|E_w^x| = |E_w^y| = \frac{|E|}{2}$ and $w_{i^*} = x$. Then $\tilde{F}(w) = x$. Let $\hat{w} \in W(2)$ be such that (1) $\hat{w}_{i^*} = y$ and (2) $\hat{w}_i = w_i$ for every $i \in E\backslash\{i^*\}$. Then $|E_{\hat{w}}^y| > \frac{|E|}{2}$ and $\tilde{F}(\hat{w}) = y$. Then, $\tilde{F}(w) \neq \tilde{F}(\hat{w})$ and there is no $i \in E$ such that $F(w) = w_i$ and $w_i\hat{w}_i \in I_i$. Therefore, by Lemma 4, $\tilde{F}$ is not implementable in Nash equilibrium. ■

PROOF OF THEOREM 6. Note that, if $\delta > 2$, then $|E| \geq 3$ and $|N| \geq 3$. Given any jury configuration I, let $F \in \mathcal{F}^M(\delta)$ be implementable in Nash equilibrium.

Case 1. $|E|$ is an odd number.

Let $w \in W(\delta)$ be such that, for some $x, y, z \in N$, $|E_w^x| = \left\lfloor \frac{|E|}{2} \right\rfloor$, $|E_w^y| = \left\lfloor \frac{|E|}{2} \right\rfloor$, and $|E_w^z| = 1$. Next, we show that $F(w) \neq x$. Suppose on the contrary that $F(w) = x$. Let $i \in E$ be such that $w_i = z$. Let $\hat{w} \in W(\delta)$ be such that (1) $\hat{w}_i = y$ and (2) $\hat{w}_j = w_j$ for every $j \in E\backslash\{i\}$. Note that $|E_{\hat{w}}^y| > \frac{|E|}{2}$. Because $F \in \mathcal{F}^M(\delta)$, $F(\hat{w}) = y$. Then, $w, \hat{w} \in W(\delta)$ are such that $F(w) \neq F(\hat{w})$. However, there is no $j \in E$ such that $F(w) = w_j$ and $w_j\hat{w}_j \in I_i$ (note that i is the only expert for whom $w_i \neq \hat{w}_i$). Hence, by Lemma 4, F is not implementable in Nash equilibrium, which is a contradiction. Using a symmetrical argument it can be proved that $F(w) \neq y$. Now, we prove that $F(w) \neq z$. Suppose on the contrary that $F(w) = z$. Let $i \in E$ be such that $w_i = y$. Let $\hat{w} \in W(\delta)$ be such that (1) $\hat{w}_i = x$ and (2) $\hat{w}_j = w_j$ for every $j \in E\backslash\{i\}$. Note that $|E_{\hat{w}}^x| > \frac{|E|}{2}$. Because $F \in \mathcal{F}^M(\delta)$, $F(\hat{w}) = x$. Then, $F(w) \neq F(\hat{w})$ and there is no $j \in E$ such that $F(w) = w_j$ and $w_j\hat{w}_j \in I_i$. By Lemma 4, this contradicts that F is implementable in Nash equilibrium.

Using a similar argument, it can be proved that there is no $t \in N \backslash \{x, y, z\}$ such that $F(w) = t$, which is a contradiction.

Case 2. $|E|$ is an even number.

Let $w \in W(\delta)$ be such that, for some $x, y, z \in N$, $|E_w^x| = \frac{|E|}{2}$, $|E_w^y| = \left\lceil \frac{|E|}{4} \right\rceil$, and $E_w^z = \left\lfloor \frac{|E|}{4} \right\rfloor$. Next, we prove that $F(w) = x$. Suppose on the contrary that $F(w) \neq x$. Let $i \in E$ be such that $w_i \neq F(w)$ and $w_i \neq x$. Let $\hat{w} \in W(\delta)$ be such that (1) $\hat{w}_i = x$ and (2) $\hat{w}_j = w_j$ for every $j \in E \backslash \{i\}$. Note that $|E_{\hat{w}}^x| > \frac{|E|}{2}$. Because $F \in \mathcal{F}^M(\delta)$, $F(\hat{w}) = x$. Then, $F(w) \neq F(\hat{w})$ and there is no $j \in E$ such that $F(w) = w_j$ and $w_j \hat{w}_j \in I_j$. By Lemma 4, this contradicts that F is implementable in Nash equilibrium. Let $\tilde{w} \in W(\delta)$ be such that, for each $i \in E$, (1) if $w_i = x$ then $\tilde{w}_i \in \{x, z\}$, (2) if $w_i = y$ then $\tilde{w}_i = y$, (3) if $w_i = z$ then $\tilde{w}_i = y$, and (4) $|E_{\tilde{w}}^x| = \left\lceil \frac{|E|}{4} \right\rceil$ and $E_{\tilde{w}}^z = \left\lfloor \frac{|E|}{4} \right\rfloor$. Note that $|E_{\tilde{w}}^y| = \frac{|E|}{2}$, $|E_{\tilde{w}}^x| = \left\lceil \frac{|E|}{4} \right\rceil$, and $E_{\tilde{w}}^z = \left\lfloor \frac{|E|}{4} \right\rfloor$. Using the same argument than above, we can prove that $F(\tilde{w}) = y$. Let $\bar{w} \in W(\delta)$ be such that, for each $i \in E$, (1) if $w_i = z$ then $\bar{w}_i = y$ and (2) if $w_i \neq z$ then $\bar{w}_i = w_i$. Note that $|E_{\bar{w}}^x| = \frac{|E|}{2}$, $|E_{\bar{w}}^y| = \frac{|E|}{2}$. Because $F(w) = x$ and there is no $i \in E$ such that $F(w) = w_i$ and $w_i \neq \bar{w}_i$, by Lemma 4 we have $F(\bar{w}) = x$. Similarly, because $F(\tilde{w}) = y$ and there is no $i \in E$ such that $F(\tilde{w}) = \tilde{w}_i$ and $\tilde{w}_i \neq \bar{w}_i$, by Lemma 4 we have $F(\bar{w}) = y$, which is a contradiction. ■

References

Amorós, P. (2013): Picking the winners, Int. J Game Theory, 42, 845-865.

Amorós, P. (2009): Eliciting socially optimal rankings from unfair jurors, J. Econ. Theory, 144, 1211-1226.

Austen-Smith, D. and Banks J.S. (1996): Information aggregation, rationality and the Condorcet jury theorem, Amer. Polit. Sci. Rev., 90, 34-45.

Crès, H., Gilboa, I., and Vieille, N. (2011): Aggregation of multiple prior opinions, J. Econ. Theory, 146, 2563-2582.

Duggan, J. and Martinelli, C. (2001): A Bayesian model of voting in juries, Games Econ. Behav., 37, 259–294.

Feddersen, T. and Pesendorfer, W. (1998): Convicting the innocent: The inferiority of unanimous jury verdict, Amer. Polit. Sci. Rev., 92, 23-35.

Gadjos, T. and Vergnaud J-C. (2013): Decisions with conflicting and imprecise information, Soc. Choice Welfare, 41, 427-452.

Gerardi, D., McLean, R., and Postlewaite, A. (2009): Aggregation of expert opinions, Games Econ. Behav., 65, 339-371.

Krishna, V. and Morgan, J. (2001): A model of expertise, Quart. J. Econ., 116, 747-775.

List, C. and Pettit, P. (2002): Aggregating sets of judgements: An impossibility result, Econ. Philos., 18, 89-110.

Maskin, E. (1999): Nash equilibrium and welfare optimality, Rev. Econ. Stud., 66, 23-38.

McLennan, A. (1998): Consequences of the Condorcet jury theorem for beneficial information aggregation by rational agents, Amer. Polit. Sci. Rev., 92,413–418.

Pettit, P. (2001): Deliberative democracy and the discursive dilemma, Philos. Issues, 11, 268-299.

Wolinsky, A. (2002): Eliciting information from multiple experts, Games Econ. Behav., 41, 141-60.