Málaga Economic Theory Research Center Working Papers

Juan D. Moreno-Ternero

Voting over piece-wise linear tax methods[*]

Juan D. Moreno-Ternero[†]

January 12, 2010

Abstract

We analyze the problem of choosing the most appropriate method for apportioning taxes in a democracy. We consider a simple model of taxation and restrict our attention to piece-wise linear tax methods, which are almost ubiquitous in advanced democracies worldwide. In spite of facing an impossibility result saying that if we allow agents to vote for any piece-wise linear tax method no equilibrium exists, we show that if we limit the domain of admissible methods in a meaningful way, albeit not restrictive, an equilibrium does exist. We also show that, for such a domain, a wide variety of methods can be supported in equilibrium. This last result provides rationale for some activities of special interest groups.

JEL numbers: D72, H24.

Keywords: voting, taxes, majority, single crossing, special interest politics.

1   Introduction

The primary struggle among citizens in all advanced democracies is over the distribution of economic resources. Income taxation, besides being a major source of state funds, is one of the essential tools for solving such a struggle, which makes it a matter of concern for politicians and economists alike. The search for the perfect income tax structure is (and has been for a long time) a milestone and even though some consensus has been reached (e.g., almost all countries in the world use statutory tax schedules specified only in terms of the brackets and

---

[*]Acknowledgment will be added later.

[†]Universidad de Málaga, Universidad Pablo de Olavide and CORE, Université catholique de Louvain.

tax rates) the discussion is far from being over.[1] In this paper, we approach this issue from a political economy perspective, upon studying the political process in which tax methods are either chosen directly by voters, according to majority rule, or via elections in a perfectly representative democracy.

Academic interest in this area started to emerge after Foley (1967), who analyzed the problem of voting over taxes in an endowment economy. Foley focused on the case of flat taxes (with or without exemption; and allowing or excluding for the existence of negative taxes) and showed that, for such a class, there always exists a majority rule equilibrium, i.e., a (flat) tax method that cannot be overturned by any other member of the class through majority rule.[2]

In this paper, we plan to focus on the class of piece-wise linear tax methods (rather than flat taxes) which, as mentioned above, seems to be almost ubiquitous in advanced democracies worldwide. For such a class, however, Foley's result does not extend and a majority rule equilibrium fails to exist. In other words, any piece-wise linear tax method can be overturned by another piece-wise linear tax method through majority rule. This is actually not more than another instance of Condorcet's paradox of voting, which is perhaps best exemplified by the problem of determining the division of a cake by majority rule (or, equivalently, tax shares by majority rule from a given initial distribution of endowments).[3]

Such a result might lead one to despair of ever achieving a voting equilibrium for any democratic polity. Nevertheless, as Campbell (1975) puts it, majority voting is never allowed to operate by itself without restraints imposed by constitution and convention. We actually show that if we limit the class of admissible methods in a meaningful way, albeit not restrictive (namely, by considering a natural subclass made of a non-countable set of piece-wise linear tax methods) the existence of a majority rule equilibrium is guaranteed. As a matter of fact, we construct the precise equilibrium for any parameter configuration of the model and show an interesting feature of it: any tax method within the class can be a majority rule equilibrium

---

[1] In the 2008 US presidential election we had a recent instance of such a discussion. President (then, Senator) Obama proposed a tax plan that would make the tax system significantly more progressive by providing large tax breaks to those at the bottom of the income scale and raising taxes significantly on upper-income earners. Senator McCain instead advocated for a tax plan that would make the tax system more regressive, upon providing relatively little tax relief to those at the bottom of the income scale while providing huge tax cuts to households at the very top of the income distribution (e.g., Burman et al., 2008).

[2] Foley's work mostly relies on verbal discussion. A more formal treatment of his model (and some of his results) is provided by Gouveia and Oliver (1996).

[3] Hamada (1973) provides a general treatment regarding why cycling is ubiquitous for this problem.

provided the predetermined level of aggregate fiscal revenue is properly chosen.

The last result mentioned above might be interpreted as a rationale for some usual tactics of special interest groups. Becker (1985) suggested that by exerting some kind of political pressure interest groups are able to affect the tax they pay. This generated a sizable literature focusing on the relationships between lobbies (special interest groups) and politicians showing, among other things, that lobbies may directly influence the policy outcome by targeting politicians (see, for instance, Grossman and Helpman (2001) and the literature cited therein). Nevertheless, such a direct channel of policy influence is not always effective, or feasible, for lobbies. An instance of this is precisely the case of direct democracy, in which politicians are simply not the policy-makers and, therefore, lobbies need to target voters. Our result provides an open window for lobbies to affect the policy outcome (in this case, the tax method) in a subtle and indirect way. Instead of lobbying politicians (or voters) directly to achieve a desired tax method, they can achieve the same goal upon lobbying for a precise aggregate fiscal revenue.

The result might also be interpreted as a rationale for the behavior of corrupt politicians that might manipulate the budget (and fix a predetermined value of the aggregate fiscal revenue) in order to lead the citizenry to a given tax schedule. Political corruption has been a persistent phenomenon throughout history and across societies. It is found today in many different forms and degrees in all types of political systems. In a well-functioning democracy, citizens hold politicians accountable for their performance. This is predicated upon voters having access to the information that allows them to evaluate politician performance (e.g., Ferraz and Finan, 2008). For this reason, a corrupt politician usually seeks to conceal his illegal activities from his constituents hence keeping voters ignorant or at least uncertain about them.[4] Our result provides a plausible option for a corrupt politician to keep voters ignorant about the goal of favoring a precise tax method by means of focusing on setting a precise aggregate fiscal revenue.

The rest of the paper is organized as follows. In Section 2, we introduce the model. In Section 3, we provide our results regarding the existence of majority-rule equilibria for piecewise linear tax methods. In Section 4, we explicitly construct the equilibria and focus on some of their properties providing the rationale for some tactics of special interest groups, as well as corrupt politicians, described above. In Section 5, we explore how to extend our analysis to the case of a model of a perfectly representative democracy. Finally, Section 6 concludes.

---

[4]Here we interpret as corrupt a political act which violates the public interest for private interest or gain. An alternative view, in contrast with the one we endorse here, holds that a political act is corrupt if the public deems it so (e.g., Rundquist et al., 1977).

2　The model

We study taxation problems in a variable population model, first introduced by Young (1988). The set of potential taxpayers, or *agents*, is identified by the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. For each $i \in N$, let $y_i \in \mathbb{R}_+$ be i's (taxable) *income* and $y \equiv (y_i)_{i \in N}$ the income profile. A (taxation) *problem* is a triple consisting of a population $N \in \mathcal{N}$, an income profile $y \in \mathbb{R}_+^N$, and a tax revenue $T \in \mathbb{R}_+$ such that $\sum_{i \in N} y_i \geq T$. Let $Y \equiv \sum_{i \in N} y_i$. To avoid unnecessary complication, we assume $Y = \sum_{i \in N} y_i > 0$. Let $\mathcal{D}^N$ be the set of taxation problems with population N and $\mathcal{D} \equiv \cup_{N \in \mathcal{N}} \mathcal{D}^N$.

Given a problem $(N, y, T) \in \mathcal{D}$, a *tax profile* is a vector $x \in \mathbb{R}^N$ satisfying the following three conditions: (i) for each $i \in N$, $0 \leq x_i \leq y_i$, (ii) $\sum_{i \in N} x_i = T$ and (iii) for each $i, j \in N$, $y_i \geq y_j$ implies $R_i(N, y, T) \geq R_j(N, y, T)$ and $y_i - R_i(N, y, T) \geq y_j - R_j(N, y, T)$. We refer to (i) as *boundedness*, (ii) as *balancedness* and (iii) as *order preservation*. A (taxation) *method* on $\mathcal{D}$, $R \colon \mathcal{D} \to \cup_{N \in \mathcal{N}} \mathbb{R}^N$, associates with each problem $(N, y, T) \in \mathcal{D}$ a tax profile $R(N, y, T)$ for the problem.[5] Instances of methods are the *head tax*, which distributes the tax burden equally, provided no agent ends up paying more than her income, the *leveling tax*, which equalizes post-tax income across agents, provided no agent is subsidized and the *flat tax*, which equalizes tax rates across agents.

All these methods are instances of *piece-wise* linear tax methods. Formally, a piece-wise linear tax method is a method associated to a vector of brackets, rates and lump-sum levies. For each bracket, a given marginal tax rate is proposed and the corresponding lump-sum levies of the brackets are designed so that the schedule moves continuously from one bracket to another. More precisely, a method R is piece-wise linear if for each $(N, y, T) \in \mathcal{D}$ there exist sequences $\{\alpha_j, \beta_j, \lambda_j\}_{j=1}^k$ such that

(i) For each $j = 1, \ldots, k$, $\alpha_j, \lambda_j \in \mathbb{R}_+$ and $\beta_j \in \mathbb{R}$;

(ii) For each $j = 1, \ldots, k-1$, $\lambda_j \leq \lambda_{j+1}$;

---

[5]In essence, the problem under consideration is a distribution problem, in which the total amount to be distributed is exogenous, and the issue is to determine methods providing an allocation for each admissible problem. There is another branch of the taxation literature in which no reference to the amount of revenue to be raised is made (e.g., Ok, 1995; Mitra and Ok, 1996). In such a branch, the basic problem is to determine a tax function yielding the tax associated to each positive income level. An underlying assumption of the corresponding models is to assume the existence of a continuum of agents (a reasonable assumption only in the case of arbitrary large populations), which allows the use of calculus. A more general approach encompassing both possibilities (namely, discrete and continuous populations) is taken by Le Breton et al., (1996).

(iii) For each $j = 1, \ldots, k$, $0 \leq \alpha_j \leq 1$.

(iv) For each $j = 1, \ldots, k-1$, $\alpha_j \lambda_j + \beta_j = \alpha_{j+1} \lambda_j + \beta_{j+1}$;

(v) For each $j = 2, \ldots, k$, $(1 - \alpha_j)\lambda_{j-1} \geq \beta_j \geq -\alpha_j \lambda_{j-1}$;

and, for each $i \in N$,

$$R_i (N, y, T) = \alpha_j y_i + \beta_j,$$

where j is such that $\lambda_{j-1} \leq y_i \leq \lambda_j$.

Note that item (iii) above guarantees that every tax schedule has slope less than one. Item (iv) guarantees that the path of taxes generated by the method, for a given level of tax revenue, is continuous. Finally, item (v) guarantees that the tax payed by each agent is neither negative nor higher than her pre-tax income.
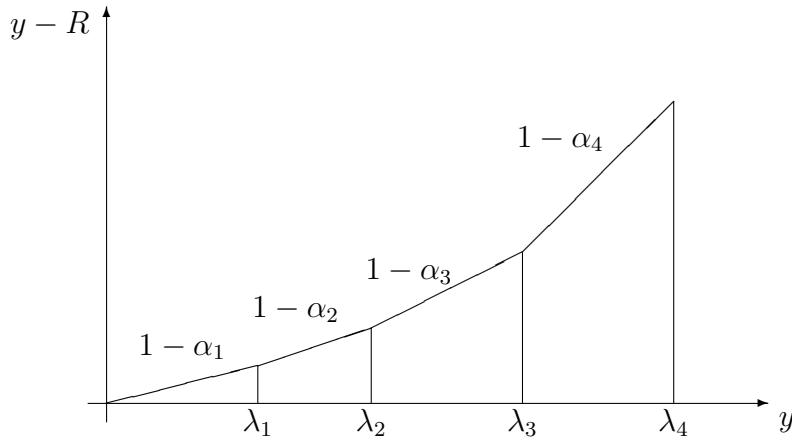
Figure 1: A piece-wise linear tax method . This figure illustrates a piece-wise linear tax method originated by $\{\alpha_j, \beta_j, \lambda_j\}_{j=1}^4$. If y is lower than λ_1, the marginal tax rate is α_1. For $\lambda_1 \leq y \leq \lambda_2$, the marginal tax rate is α_2. For $\lambda_2 \leq y \leq \lambda_3$, the marginal tax rate is α_3, and for $\lambda_3 \leq y \leq \lambda_4$, the marginal tax rate is α_4.

We will analyze the problem in which agents vote for tax methods according to majority rule. We assume that voters are self-interested: given a pair of alternatives, a taxpayer votes for the alternative that gives her the greatest post-tax income. We say that a method R is a majority rule equilibrium for a set of methods $\mathcal{S}$ if, for any $(N, y, T) \in \mathcal{D}$, there is no other method $R' \in \mathcal{S}$ such that, $y - R'(N, y, T) > y - R(N, y, T)$ for the majority of voters.

3 The existence of equilibrium

3.1 A negative result

We start this section with a negative result.

Theorem 1 *There is no majority rule equilibrium for the family of piece-wise linear tax methods.*

Even though the technical proof of this result might be cumbersome, its logic should be clear. It all amounts to realize that given a piece-wise linear tax method, one can construct another (piece-wise linear) method increasing taxes for a small group of taxpayers and reducing the burden for all the others, while keeping the tax revenue constant (see Figure 2). The argument, which is even valid for two-piece linear methods, is similar to others used in related models (e.g., Hamada, 1973; Marhuenda and Ortuño-Ortín, 1998).

A caveat is worth mentioning. If more than half of the agents are paying zero taxes, we cannot reduce their burdens and thus the corresponding tax allocation could not be defeated through majority rule by any other allocation. Nevertheless, there is no method guaranteeing that more than half of the agents are paying zero taxes for any admissible problem (although there certainly exist methods doing so for specific problems). The most extreme case would be the leveling tax, which would always be the most preferred method by the agent with the lowest income. This method, however, can be defeated by other piece-wise linear methods in many problems (in which, needless to say, there is not a majority of the population facing a zero tax burden with the leveling tax).
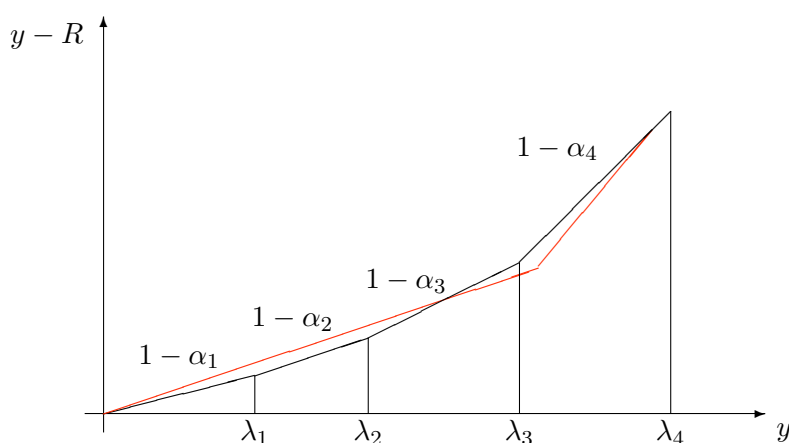
Figure 2: A piece-wise linear tax method defeating another. This figure illustrates a piece-wise linear tax method defeating a given one upon reducing the burden for low-income agents and increasing it for a small group of high-income agents.

3.2 A positive result

Given Theorem 1 our aim now shifts to prove the existence of a majority rule equilibrium for a sufficiently large family of piece-wise linear tax methods. To do so, let us consider the following family $\left(\{R^\theta\}_{\theta \in [0,1]}\right)$ made of a continuum of piece-wise linear methods and referred to as the *TAL-family* of methods.[6]

Fix $\theta \in [0,1]$. For all $(N, y, T) \in \mathcal{D}$, and all $i \in N$,

$$R_i^\theta (N, y, T) = \begin{cases} \min\{\theta y_i, \lambda\} & \text{if } T \leq \theta Y \\ \max\{\theta y_i, y_i - \mu\} & \text{if } T \geq \theta Y \end{cases}$$

where λ and μ are chosen so that $\sum_{i \in N} R_i^\theta (N, y, T) = T$.

The tax methods in this family have two components. The first component is a proportional contribution given by the value of the parameter θ. The second component is a term that depends on whether the amount of taxes collected by this proportional system exceeds or falls short of what is required. More precisely, $R_i^\theta (N, y, T) = \theta y_i + \beta_i (N, y, T)$ for all $i \in N$, where the precise formula of $\beta_i (N, y, T)$ depends on the difference between T and θY. If $T = \theta Y$ the tax burden is allocated as with the flat tax, i.e., $\beta_i (N, y, T) = 0$ for all $i \in N$. If $T < \theta Y$, (i.e., the proportional tax collection exceeds the tax burden) the richest agents get a rebate of $\beta_i (N, y, T) = \lambda - \theta y_i$ so that they end up paying the same amount. That is to say, they are taxed marginally at zero. If, however, $T > \theta Y$ (i.e., the proportional tax collection is insufficient to cover the needs) the richest agents will increase their contributions with $\beta_i (N, y, T) = (1 - \theta)y_i - \mu$, which thus represents an overall marginal tax rate of 1.[7]

Clearly, any given problem (N, y, T) will fall within the last case by reducing sufficiently the value of the parameter θ. In the limit, i.e., for $\theta = 0$, the tax method corresponds to the leveling tax. Symmetrically, by letting $\theta \to 1$, we can always make of this taxation problem one of the second case. In the limit, i.e., $\theta = 1$, the tax method coincides with the head tax.

The methods in the family impose to each taxpayer a rationing of the same sort as that faced by the whole society. Namely, if the tax burden is below a certain fraction of the aggregate income, then no taxpayer can pay more than such a fraction of her gross income. Similarly, if the burden is above a certain fraction of the aggregate income, then no taxpayer can pay less than such a fraction of her gross income.

---

[6]This family was introduced, in the dual framework of bankruptcy problems, by Moreno-Ternero and Villar (2006).

[7]Note that "the rich" in both cases are endogenously determined as a function of T, θ and the whole income distribution vector y.
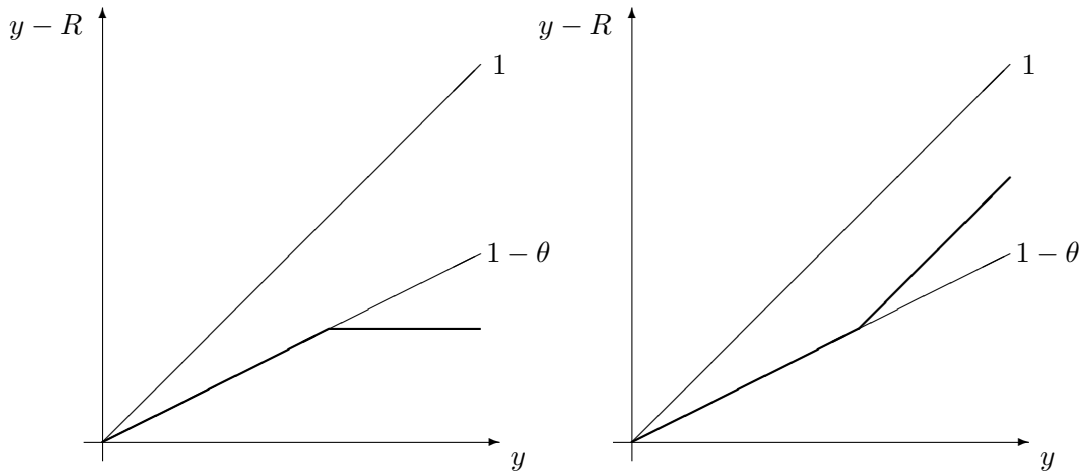
Figure 3: The tax method R^θ. This figure illustrates the two possible types of tax schemes originated by R^θ to yield a predetermined tax return T. If T is relatively small $(T \leq \theta Y)$, the marginal tax rate is θ up to some income level, and 0 afterwards. If X is relatively large $(T \geq \theta Y)$, the marginal tax rate is θ and then 1.

The family is indeed a generalization of a method inspired in the Talmud (e.g., Aumann and Maschler, 1985), which corresponds to the case in which the fraction of the aggregate income in the above definition is precisely one half, which can be regarded as a psychological threshold appropriated to switch the focus from taxes to net incomes. Moving such a threshold to any arbitrary fraction, we obtain a wide variety of options. As a matter of fact, we can see how the family encompasses a whole continuum set of methods ranging from the "least" progressive (the *needs-blind* head tax) to the "most" progressive (the *incentives-blind* leveling tax) piece-wise linear tax schemes.[8] Thus, voters are confronted with a wide variety of choices to select the best tax scheme, even if we restrict their options to this family. Restricting to a one-parameter family of tax methods in which the parameter reflects the degree of progressivity (or regressivity) of the method is a usual course of action in taxation models (e.g., Bénabou, 2000, 2002).

The methods in the TAL-family obey two natural principles in taxation. The first one, known as *consistency*, says that the way that taxpayers split a given tax total depends only on their own taxable incomes. In other words, consistency expresses the robustness of a method under the departure of some agents with their contributions. The second one, known as *scale invariance*, says that the relative distribution of taxes should not depend on anything more than the relative sizes of taxable incomes.

---

[8]It is not difficult to show that the head (resp. leveling) tax shares the tax burden so that the corresponding post-tax income allocation is less (resp. more) equitable than the post-tax income allocation provided by any other method.

It turns out that the TAL-family comprises all the consistent and scale-invariant methods that share a pattern of equality in the tax profiles they propose (e.g., Thomson, 2008).

Finally, it is worth mentioning that even though the flat tax is not a member of the TAL-family, all tax profiles proposed by such method are covered by the family. More precisely, if F denotes the flat tax method, then, for each $(N, y, T) \in \mathcal{D}$, $F(N, y, T) = R^{\theta}(N, y, T)$, for $\theta = \frac{T}{Y}$.

The main result of this section is the following:

Theorem 2 *There is a majority rule equilibrium for the TAL-family of tax methods.*

In order to prove Theorem 2, we need the following lemma regarding the TAL-family, which is interesting on its own, and whose proof appears in the appendix.

Lemma 1 *Let $0 \leq \theta_1 \leq \theta_2 \leq 1$ and $(N, y, T) \in \mathcal{D}$. If n denotes the agent in N with the highest income then $R_n^{\theta_1}(N, y, T) \geq R_n^{\theta_2}(N, y, T)$.*

We also need to introduce the following concept:

*A method R **single-crosses** R' if for each $(N, y, T) \in \mathcal{D}$, we have the following:*

(i) If $R_i(N, y, T) \leq R_i'(N, y, T)$ then $R_j(N, y, T) \leq R_j'(N, y, T)$ for all j such that $y_j \leq y_i$ and

(ii) If $R_i(N, y, T) \geq R_i'(N, y, T)$ then $R_j(N, y, T) \geq R_j'(N, y, T)$ for all j such that $y_j \geq y_i$.

The single-crossing property allows one to separate those agents who benefit from the application of one method or the other, depending on the rank of their incomes. That is, if the methods R and R' satisfy this property and $R_i(N, y, T) \leq R_i'(N, y, T)$ for some i, then R' will give higher or equal shares to all agents with incomes smaller than y_i. Similarly, if $R_i(N, y, T) \geq R_i'(N, y, T)$ for some i, then R will give higher or equal shares to all agents with incomes higher than y_i.

It is well known that a sufficient condition for the existence of a majority rule equilibrium is that voters exhibit intermediate preferences over the set of alternatives (e.g., Gans and Smart, 1996). Thus, since we assume that voters are self-interested and therefore simply vote according to the post-tax incomes that methods offer to them, it suffices to show that, for any pair of values $\theta_1, \theta_2 \in [0, 1]$, R^{θ_1} single-crosses R^{θ_2}. To do so, let $\theta_1 \leq \theta_2 \in [0, 1]$ and $(N, y, T) \in \mathcal{D}$ be given. For ease of exposition, assume that $N = \{1, \ldots, n\}$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Then, it is enough to show that there exists some $i^* \in N$ such that:

(i) $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., i^*$ and

(ii) $R_i^{\theta_1}(N, y, T) \geq R_i^{\theta_2}(N, y, T)$ for all $i = i^* + 1, ..., n$.

To do so, we distinguish three cases:

Case 1: $T \leq \theta_1 Y$.

By the definition of the TAL-family, $R_i^{\theta_j}(N, y, T) = \min\{\theta_j y_i, \lambda_j\}$, for all $i \in N$ and $j = 1, 2$, where λ_1 and λ_2 are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0, ..., n\}$ such that $T \leq \theta_1((\sum_{i=1}^{r_1} y_i) + (n - r_1)y_{r_1+1})$ and r_2 the smallest non-negative integer in $\{0, ..., n\}$ such that $T \leq \theta_2((\sum_{i=1}^{r_2} y_i) + (n - r_2)y_{r_2+1})$. Note that it is straightforward to show that $r_2 \leq r_1$. Thus,

$$\begin{aligned}
R^{\theta_1}(N, y, T) &= (\theta_1 y_1, ..., \theta_1 y_{r_2}, ..., \theta_1 y_{r_1}, \lambda_1, ..., \lambda_1), \text{ and} \\
R^{\theta_2}(N, y, T) &= (\theta_2 y_1, ..., \theta_2 y_{r_2}, \lambda_2, ..., \lambda_2, ..., \lambda_2),
\end{aligned}$$

where $\lambda_1 = \frac{T - \theta_1(\sum_{i=1}^{r_1} y_i)}{n - r_1}$ and $\lambda_2 = \frac{T - \theta_2(\sum_{i=1}^{r_2} y_i)}{n - r_2}$. Consequently, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., r_2$ and, by Lemma 1, $R_i^{\theta_1}(N, y, T) \geq R_i^{\theta_2}(N, y, T)$ for all $i = r_1 + 1, ..., n$. Now, there are three subcases:

Subcase 1.1: $\lambda_2 < \theta_1 y_{r_2+1}$.

Then, $i^* = r_2 + 1$ and the single-crossing property holds.

Subcase 1.2: $\lambda_2 \geq \theta_1 y_{r_1}$.

Then, $i^* = r_1 + 1$ and the single-crossing property holds.

Subcase 1.3: $\lambda_2 \in [\theta_1 y_{r_2+1}, \theta_1 y_{r_1}]$.

Then, there exists some $k \in \{r_2 + 1, ..., r_1 - 1\}$ such that $\theta_1 y_{k+1} > \lambda_2 \geq \theta_1 y_k$. Thus, $i^* = k$ and the single-crossing property holds.

Case 2: $T \geq \theta_2 Y$.

By the definition of the TAL-family, $R_i^{\theta_j}(N, y, T) = \max\{\theta_j y_i, y_i - \mu_j\}$, for all $i \in N$ and $j = 1, 2$, where μ_1 and μ_2 are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0, ..., n\}$ such that $T \geq \theta_1 Y + (1 - \theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n - r_1)y_{r_1+1})$.[9] Furthermore, let r_2 be the smallest non-negative integer in $\{0, ..., n\}$ such that $T \geq \theta_2 Y + (1 - \theta_2)((\sum_{i=r_2+1}^{n} y_i) - (n - r_2)y_{r_2+1})$. Note that it is straightforward to show that $r_2 \leq r_1$. Thus,

$$\begin{aligned}
R^{\theta_1}(N, y, T) &= (\theta_1 y_1, ..., \theta_1 y_{r_2}, ..., \theta_1 y_{r_1}, y_{r_1+1} - \mu_1, ..., y_n - \mu_1), \text{ and} \\
R^{\theta_2}(N, y, T) &= (\theta_2 y_1, ..., \theta_2 y_{r_2}, y_{r_2+1} - \mu_2, ..., y_n - \mu_2),
\end{aligned}$$

where $\mu_1 = \frac{\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{n} y_i) - T}{n - r_1}$ and $\mu_2 = \frac{\theta_2(\sum_{i=1}^{r_2} y_i) + (\sum_{i=r_2+1}^{n} y_i) - T}{n - r_2}$. By Lemma 1, $R_n^{\theta_1}(N, y, T) \geq R_n^{\theta_2}(N, y, T)$. Thus, $\mu_1 \leq \mu_2$. Consequently, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., r_2$ and $R_i^{\theta_1}(N, y, T) \geq R_i^{\theta_2}(N, y, T)$ for all $i = r_1 + 1, ..., n$. Now, there are three subcases:

Subcase 2.1: $\mu_2 < (1 - \theta_1)y_{r_2+1}$.

Then, $i^* = r_1$ and the single-crossing property holds.

Subcase 2.2: $\mu_2 \geq (1 - \theta_1)y_{r_1}$.

---

[9]For the sake of completeness, assume $y_0 = 0$.

Then, $i^* = r_2$ and the single-crossing property holds.

Subcase 2.3: $\mu_2 \in [(1 - \theta_1)y_{r_2+1}, (1 - \theta_1)y_{r_1}]$.

Then, there exists some $k \in \{r_2 + 1, ..., r_1 - 1\}$ such that $(1 - \theta_1)y_{k+1} > \mu_2 \geq (1 - \theta_1)y_k$. Thus, $i^* = k$ and the single-crossing property holds.

Case 3: $\theta_1 Y < T < \theta_2 Y$.

By the definition of the TAL-family, $R_i^{\theta_1}(N, y, T) = \max\{\theta_1 y_i, y_i - \mu\}$ and $R_i^{\theta_2}(N, y, T) = \min\{\theta_2 y_i, \lambda\}$ for all $i \in N$, where μ and λ are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0, ..., n - 1\}$ such that $T \geq \theta_1 Y + (1 - \theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n - r_1)y_{r_1+1})$. Furthermore, let r_2 be the smallest non-negative integer in $\{0, ..., n - 1\}$ such that $T \leq \theta_2((\sum_{i=1}^{r_2} y_i) + (n - r_2)y_{r_2+1})$. Thus,

$$R^{\theta_1}(N, y, T) = (\theta_1 y_1, ..., \theta_1 y_{r_1}, y_{r_1+1} - \mu, ..., y_n - \mu), \text{ and}$$

$$R^{\theta_2}(N, y, T) = (\theta_2 y_1, ..., \theta_2 y_{r_2}, \lambda, ..., \lambda),$$

where $\lambda = \frac{T - \theta_2(\sum_{i=1}^{r_2} y_i)}{n - r_2}$ and $\mu = \frac{\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^{n} y_i) - T}{n - r_1}$. Consequently, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., \min\{r_1, r_2\}$. We distinguish two subcases.

Subcase 3.1: $r_1 \geq r_2$.

Then, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., r_2$. By Lemma 1, $R_n^{\theta_1}(N, y, T) \geq R_n^{\theta_2}(N, y, T)$. Let k be the smallest non-negative integer in N such that $R_k^{\theta_1}(N, y, T) \geq R_k^{\theta_2}(N, y, T)$.[10] Two options are then open. If $k \geq r_1 + 1$, then $y_k - \mu = R_k^{\theta_1}(N, y, T) \geq R_k^{\theta_2}(N, y, T) = \lambda$. Thus, $y_{k'} \geq \mu + \lambda$ for all $k' = k, ..., n$, or equivalently, $R_{k'}^{\theta_1}(N, y, T) \geq R_{k'}^{\theta_2}(N, y, T)$ for all $k' = k, ..., n$ and the single-crossing property follows. If, on the other hand, $r_2 + 1 \leq k \leq r_1$, then $\theta_1 y_k = R_k^{\theta_1}(N, y, T) \geq R_k^{\theta_2}(N, y, T) = \lambda$. Thus, $\theta_1 y_{k'} \geq \lambda$ for all $k' = k, ..., r_1$, or equivalently, $R_{k'}^{\theta_1}(N, y, T) \geq R_{k'}^{\theta_2}(N, y, T)$ for all $k' = k, ..., r_1$. Now, since $R_{r_1+1}^{\theta_1}(N, y, T) = y_{r_1+1} - \mu = \max\{\theta_1 y_{r_1+1}, y_{r_1+1} - \mu\}$ we know that $\mu \leq (1 - \theta_1)y_{r_1+1}$. Furthermore, since $\theta_1 y_{r_1} \geq \lambda$ we obtain $\mu + \lambda \leq (1 - \theta_1)y_{r_1+1} + \theta_1 y_{r_1} \leq y_{r_1+1} \leq y_{k'}$ for all $k' = r_1 + 1, ..., n$. As a result, $R_{k'}^{\theta_1}(N, y, T) \geq R_{k'}^{\theta_2}(N, y, T)$ for all $k' = k, ..., n$, and the single-crossing property follows.

Subcase 3.2: $r_1 < r_2$.

Then, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., r_1$. Furthermore, by Lemma 1, $R_n^{\theta_1}(N, y, T) \geq R_n^{\theta_2}(N, y, T)$. Let k be the smallest non-negative integer in N such that $R_k^{\theta_1}(N, y, T) \geq R_k^{\theta_2}(N, y, T)$. As before, we have two options. If $k \geq r_2 + 1$, then $y_k - \mu = R_k^{\theta_1}(N, y, T) \geq R_k^{\theta_2}(N, y, T) = \lambda$. Thus, $y_{k'} \geq \mu + \lambda$ for all $k' = k, ..., n$, or equivalently, $R_{k'}^{\theta_1}(N, y, T) \geq R_{k'}^{\theta_2}(N, y, T)$ for all $k' = k, ..., n$, and the single-crossing property follows. If, on the other hand, $r_1 + 1 \leq k \leq r_2$, then, $y_k - \mu = R_k^{\theta_1}(N, y, T) \geq R_k^{\theta_2}(N, y, T) = \theta_2 y_k$. Thus, $\theta_2 y_{k'} \leq y_{k'} - \mu$ for all $k' = k, ..., r_2$, or equivalently,

---

[10]Note that $k \geq r_2$.

11

$R_{k'}^{\theta_1}(N, y, T) \geq R_{k'}^{\theta_2}(N, y, T)$ for all $k' = k, ..., r_2$. Now, since $R_{r_2+1}^{\theta_2}(N, y, T) = \lambda = \min\{\theta_2 y_{r_2+1}, \lambda\}$ we know that $\lambda \leq \theta_2 y_{r_2+1}$. Furthermore, since $\theta_2 y_{r_2} \leq y_{r_2} - \mu$ we obtain that $\mu + \lambda \leq (1-\theta_2)y_{r_2} + \theta_2 y_{r_2+1} \leq y_{r_2+1} \leq y_{k'}$ for all $k' = r_2 + 1, ..., n$. As a result, $R_{k'}^{\theta_1}(N, y, T) \geq R_{k'}^{\theta_2}(N, y, T)$ for all $k' = k, ..., n$ and the single-crossing property follows.

The proof of Theorem 2 is in that way completed. As a matter of fact, the proof tells us that the majority rule equilibrium for the TAL-family of tax methods is precisely the method preferred by the median voter, i.e., the median taxpayer.

4 Some features of the equilibrium

In what follows, we make the following mild assumption, which reflects a well-established empirical fact in advanced democracies.

Assumption 0. In each taxation problem, the median income is below the mean income.

Our next result summarizes the main findings within this section. To ease the exposition of its statement we assume, without loss of generality, that, for each $(N, y, T) \in \mathcal{D}$, $N = \{1, \ldots, n\}$ with $n \geq 3$ odd, $y_1 \leq y_2 \leq \cdots \leq y_n$, and $m = \frac{n+1}{2}$ denotes its median taxpayer. Furthermore, let $\overline{Y}^m = \sum_{j=m+1}^{n} y_j + (n-m)y_m$ and $\underline{Y}^m = \sum_{j=1}^{m} y_j + (n-m)y_m$.

Theorem 3 *If Assumption 0 holds, then the majority rule equilibrium for the TAL-family of tax methods is:*

- *The leveling tax, if $T \leq \overline{Y}^m$.*

- *The method corresponding to $\theta = \frac{T - \overline{Y}^m}{\underline{Y}^m}$, if $T \geq \overline{Y}^m$.*

Proof. We start with a piece of notation. Let $(N, y, T) \in \mathcal{D}$ be given in the conditions described above. For each $k \in N$ consider the following thresholds:

$$
\begin{aligned}
\theta_1^k &= 1 - \frac{Y - T}{n y_1}, \\
\theta_2^k &= \frac{T - \sum_{j=k+1}^{n} y_j + (n-k)y_k}{\sum_{j=1}^{k} y_j + (n-k)y_k}, \\
\theta_3^k &= \frac{T}{\sum_{j=1}^{k-1} y_j + (n-k+1)y_k}, \\
\theta_4^k &= \frac{T}{n y_1}.
\end{aligned}
$$

It is straightforward to show that $\theta_1^k \leq \theta_2^k \leq \theta_3^k \leq \theta_4^k$, and that $\theta_k^2 \leq 1$, and $\theta_k^3 > 0$. It can also be

shown that, if $0 \leq \theta_1^k \leq \theta_4^k \leq 1$,

$$R_k^\theta(N, y, T) = \begin{cases} y_k - \frac{Y-T}{n} & \text{if } \theta \leq \theta_k^1 \\ f(\theta) & \text{if } \theta_k^1 \leq \theta \leq \theta_k^2 \\ \theta y_k & \text{if } \theta_k^2 \leq \theta \leq \theta_k^3 \\ g(\theta) & \text{if } \theta_k^3 \leq \theta \leq \theta_k^4 \\ \frac{T}{n} & \text{if } \theta \geq \theta_k^4, \end{cases} \qquad (1)$$

where $f(\theta)$ is a piece-wise linear decreasing function and $g(\theta)$ is a piece-wise linear function.[11] A graphical illustration appears in Figure 4.

Let k now be the median agent, i.e., $k = m$. Then, by Assumption 0, it follows that $y_k - \frac{Y-T}{n} \leq \frac{T}{n}$. As $\theta_3^k > 0$ and $\theta_2^k < 1$, there would be nine possible cases depending of the relative positions of the remaining θ^k-thresholds with respect to 0 and 1. Nevertheless, they summarize in two supra-cases, for our purposes, in light of (1). If $\theta_2^k < 0$ then the minimum of $R_k^\theta(N, y, T)$, and therefore the most preferred method by agent k, is achieved for $\theta = 0$. If, otherwise, $\theta_2^k > 0$, then the minimum of $R_k^\theta(N, y, T)$, and therefore the most preferred method by agent k, is achieved for $\theta_2 = \theta_2^m$. This concludes the proof. ∎
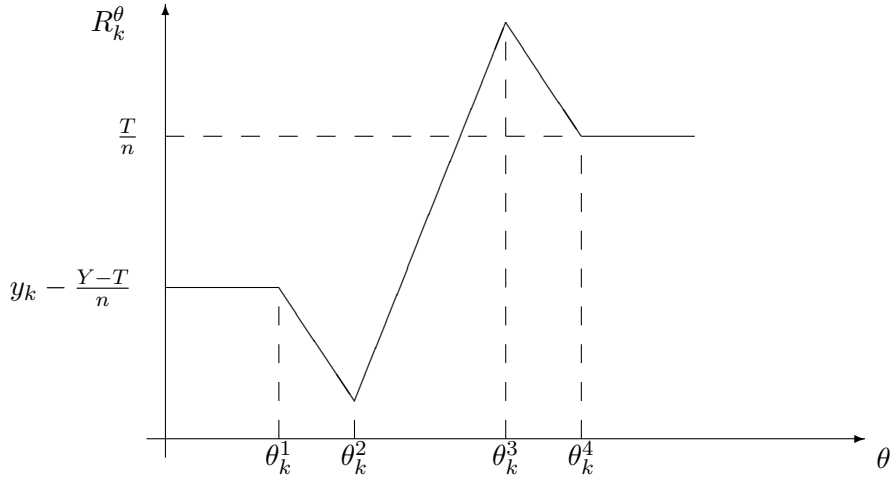
Figure 4: Individual preferences. This figure illustrates the tax burden proposed by the method R^θ for agent k as a function of the parameter θ.[12]

It is straightforward to note that, if $T = Y$, then $\frac{T - \overline{Y}^m}{\underline{Y}^m} = 1$. Thus, we have the next corollary.

Corollary 1 *Under Assumption 0, any method within the TAL-family of tax methods can be the majority rule equilibrium for this family, for a given predetermined level of aggregate fiscal revenue.*

---

[11]Note that $\theta_1^1 = \theta_2^1$ and $\theta_3^1 = \theta_4^1$, whereas $\theta_2^n = \theta_3^n$. Thus, the taxpayers with the lowest and highest incomes only have three pieces (two of them constant with respect to θ) in their preferences.

[12]For simplicity, we consider the second and fourth pieces as linear in the picture, although they are indeed piece-wise linear, as mentioned above.

Theorem 3 provides an explicit expression for the majority rule equilibrium within the TAL-family of tax methods, as a function of the data of the tax problem (namely, the group of taxpayers and the predetermined level of aggregate fiscal revenue). Corollary 1 goes further and shows that, for a given group of taxpayers, and a given method within the family, there exists a predetermined level of aggregate fiscal revenue for which such a method is the equilibrium. Thus, if there is freedom to determine the level of aggregate fiscal revenue to be raised, a given method can be targeted to become the majority rule equilibrium. Special interest groups would therefore have a strategy here to avoid the negative image they have been portrayed with. If their goal would be to implement a given tax method (within the family we are considering) they would just need to lobby for a certain level of aggregate fiscal revenue, as the will of the people would then take care of selecting their targeted method. Likewise, a corrupt government could just manipulate the budget to favor the implementation of a given method and somehow hide their move behind the veil provided by the will of the people.

5   Further insights

Our previous analysis is based on a model of direct democracy. We now explore how to extend the analysis to allow for political competition. To do so, we assume that, as in most advanced democracies, the process of political competition is organized through parties that compete in a general election. We assume that there are two parties (here denoted 1 and 2) running in this election and that competition occurs only over tax policies. Given a pair of alternative policies, a taxpayer votes for the one she prefers (i.e., the one that gives her the greatest post-tax income). If she is indifferent, she votes for each policy with probability one-half. Let $\rho(R^1, R^2)$ denote the fraction of voters that vote for party 1, when (R^1, R^2) is the pair of proposed policies. Then, the ensuing probability that party 1 wins is given by:

$$\pi(R^1, R^2) = \begin{cases} 1 & \text{if } \rho(R^1, R^2) > 0.5 \\ 0.5 & \text{if } \rho(R^1, R^2) = 0.5 \\ 0 & \text{if } \rho(R^1, R^2) > 0.5 \end{cases}$$

Finally, let $\Pi^i(R^1, R^2)$ denote the payoff or utility that party $i = 1, 2$ gets when that pair of policies is proposed by both parties. We then assume that

$$\Pi^1(R^1, R^2) = \pi(R^1, R^2) \text{ and } \Pi^2(R^1, R^2) = 1 - \pi(R^1, R^2).$$

A *political equilibrium* will be a Nash equilibrium of the resulting game played by the two parties, where the payoff functions are described as above and they share a common policy space. We have the following result, in which we assume the notational convention of the previous section.

Theorem 4 *If the policy space for both parties is the TAL-family of tax methods, and Assumption 0 holds, then the political equilibrium consists of both parties proposing:*

- *The leveling tax, if $T \leq \overline{Y}^m$.*

- *The method corresponding to $\theta = \frac{T - \overline{Y}^m}{\underline{Y}^m}$, if $T \geq \overline{Y}^m$.*

Proof. Let $(N, y, T) \in \mathcal{D}$ be given and $\theta^m = \max\{0, \frac{T - \overline{Y}^m}{\underline{Y}^m}\}$.

Claim: $\pi(R^{\theta^m}, R^\theta) = 1$, for any $\theta \neq \theta^m$.

In order to prove the claim, we need three steps. Assume, without loss of generality that $\theta^m < \theta$.[13]

Step 1: $R_n^{\theta^m} > R_n^\theta$ unless $R_k^{\theta^m} = R_k^\theta$, or $y_k - R_k^{\theta^m} = y_k - R_k^\theta$, for all $k \in N$.

Step 2: $R_1^{\theta^m} < R_1^\theta$ unless $R_k^{\theta^m} = R_k^\theta$, or $y_k - R_k^{\theta^m} = y_k - R_k^\theta$, for all $k \in N$.

Step 3: $R_m^{\theta^m} = \theta^m y_m$.

This first two steps can be shown upon slightly modifying the proof of Lemma 1. The third step is a straightforward consequence of the proof of Theorem 3. The combination of the three steps implies that m, and either 1 or n, strictly prefer the method R^{θ^m} to the method R^θ. From here, the single-crossing condition concludes the proof of the claim.

The claim guarantees that both parties playing the method R^{θ^m} is indeed a political equilibrium as a party deviating from that profile would decrease its probability of victory from 0.5 to 0. The claim also guarantees uniqueness. If a party is not playing R^{θ^m}, but its opponent is, it will have an incentive to change its strategy and play it too, as that would imply increasing the probability of victory from 0 to 0.5. Finally, if none of the parties is playing R^{θ^m}, at least one of them will have an incentive to deviate and play it, as this will increase its probability of victory to 1. ∎

To conclude, we have a straightforward consequence of the last results.

Corollary 2 *Under Assumption 0, any method within the TAL-family of tax methods can be supported as the unique political equilibrium, for a given predetermined level of aggregate fiscal revenue.*

The effect of special interest groups in models of representative democracy has typically been associated to a reason why political parties stop catering to the median or average voter favoring certain types of voters over others (e.g., Grossman and Helpman, 2001). Our reading of the above corollary as a rationale for some activities of special interest groups is based on a different argument. What the corollary says is that the equilibrium policy can be altered upon moving the predetermined level of tax revenue. But this in itself is not a surprise as it only amounts to saying that the equilibrium policy depends on such predetermined level of tax revenue. The remarkable aspect is that each policy in the set could be achieved as an equilibrium, and this always being a "median voter" equilibrium.

---

[13]Otherwise, we just need to change the inequalities appearing in the proof.

Thus, parties would always be catering to the median voter, but the median voter would change depending on the predetermined level of tax revenue.

Another way of reading the above corollary is as a neutrality condition for the TAL-family of tax methods. In other words, the corollary is saying that there is no bias in favor, or against, any of the rules within the family as any of them can arise as a political equilibrium.

6   Concluding remarks

We have dealt in this paper with the issue of designing the most appropriate income tax. There is a broad consensus worldwide about implementing piece-wise linear tax methods and therefore we have endorsed such a restriction in our (simple) modeling. A key aspect regarding the implementation of a piece-wise linear tax method is the choice of the corresponding brackets, rates and lump-sum levies. Here we have analyzed such aspect assuming that the tax parameters are chosen directly by voters according to majority rule. We have provided three main results. First, an impossibility result saying that if we allow agents to vote freely for any piece-wise linear tax method, no equilibrium can come out of it. Second, we show that if we restrict the universe in a meaningful way, albeit not restrictive, an equilibrium does exist. Third, we show that, within such a (maximal) restricted domain, basically any method can be the majority rule equilibrium, upon selecting precisely the level of aggregate fiscal revenue. The results also hold for the case of a perfectly representative democracy in which tax methods arise as a result of political competition.

The restriction to piece-wise linear tax methods has not been our only assumption in the model. We have also imposed a constraint on the tax structure indicating that there is a predetermined level of aggregate fiscal revenue that has to be raised. This is a standard feature of both optimal tax models and voting models (e.g., Mirrlees, 1971; Romer, 1975; Roberts, 1977, Young, 1988). On the other hand, a standard feature in optimal tax models that has been dismissed here (as in much of the literature cited throughout this paper) is the existence of individual incentives. This is a shortcoming of our approach, in which we assume that labor is perfectly inelastically supplied. It is left for further research to extend our analysis to a more general model of taxation in which incomes would result from some type of economic choices and negative taxation (i.e., subsidies) would also be allowed. An instance of such a model has been recently studied by Fleurbaey and Maniquet (2006), who look for the optimal income tax on the basis of efficiency and fairness principles (and under incentive-compatibility constraints).[14] In their model, agents have unequal skills (and, therefore, unequal earning abilities) and heterogeneous preferences over consumption and leisure (and, therefore, unequal labour time choices). It would be interesting to explore voting situations in such a model.

---

[14]See also Fleurbaey (2006) for an extension of such analysis to the multiple commodity case.

7   Appendix

Proof of Lemma 1

We distinguish three cases:

Case 1: $T \leq \theta_1 Y$.

By the definition of the TAL-family, $R_i^{\theta_j}(N, y, T) = \min\{\theta_j y_i, \lambda_j\}$, for all $i \in N$ and $j = 1, 2$, where λ_1 and λ_2 are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0, ..., n\}$ such that $T \leq \theta_1((\sum_{i=1}^{r_1} y_i) + (n - r_1)y_{r_1+1})$ and r_2 the smallest non-negative integer in $\{0, ..., n\}$ such that $T \leq \theta_2((\sum_{i=1}^{r_2} y_i) + (n - r_2)y_{r_2+1})$. Note that it is straightforward to show that $r_2 \leq r_1$. Thus,

$$R^{\theta_1}(N, y, T) = (\theta_1 y_1, ..., \theta_1 y_{r_2}, ..., \theta_1 y_{r_1}, \lambda_1, ..., \lambda_1), \text{ and}$$
$$R^{\theta_2}(N, y, T) = (\theta_2 y_1, ..., \theta_2 y_{r_2}, \lambda_2, ..., \lambda_2, ..., \lambda_2),$$

where $\lambda_1 = \frac{T - \theta_1(\sum_{i=1}^{r_1} y_i)}{n - r_1}$ and $\lambda_2 = \frac{T - \theta_2(\sum_{i=1}^{r_2} y_i)}{n - r_2}$. Consequently, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., r_2$. Assume, by contradiction, that $R_n^{\theta_1}(N, y, T) < R_n^{\theta_2}(N, y, T)$, i.e., $\lambda_1 < \lambda_2$. Then, $R_i^{\theta_1}(N, y, T) < R_i^{\theta_2}(N, y, T)$ for all $i = r_1 + 1, ..., n$. Finally, let $k \in \{r_2 + 1, ..., r_1 - 1\}$. Then, $R_k^{\theta_1}(N, y, T) = \theta_1 y_k \leq \lambda_1 < \lambda_2 = R_k^{\theta_2}(N, y, T)$. Thus,

$$T = \sum_{i=1}^n R_i^{\theta_1}(N, y, T) < \sum_{i=1}^n R_i^{\theta_2}(N, y, T) = T,$$

which represents a contradiction.

Case 2: $T \geq \theta_2 Y$.

By the definition of the TAL-family, $R_i^{\theta_j}(N, y, T) = \max\{\theta_j y_i, y_i - \mu_j\}$, for all $i \in N$ and $j = 1, 2$, where μ_1 and μ_2 are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0, ..., n\}$ such that $T \geq \theta_1 Y + (1 - \theta_1)((\sum_{i=r_1+1}^n y_i) - (n - r_1)y_{r_1+1})$. Furthermore, let r_2 be the smallest non-negative integer in $\{0, ..., n\}$ such that $T \geq \theta_2 Y + (1 - \theta_2)((\sum_{i=r_2+1}^n y_i) - (n - r_2)y_{r_2+1})$. Note that it is straightforward to show that $r_2 \leq r_1$. Thus,

$$R^{\theta_1}(N, y, T) = (\theta_1 y_1, ..., \theta_1 y_{r_2}, ..., \theta_1 y_{r_1}, y_{r_1+1} - \mu_1, ..., y_n - \mu_1), \text{ and}$$
$$R^{\theta_2}(N, y, T) = (\theta_2 y_1, ..., \theta_2 y_{r_2}, y_{r_2+1} - \mu_2, ..., y_n - \mu_2),$$

where $\mu_1 = \frac{\theta_1(\sum_{i=1}^{r_1} y_i) + (\sum_{i=r_1+1}^n y_i) - T}{n - r_1}$ and $\mu_2 = \frac{\theta_2(\sum_{i=1}^{r_2} y_i) + (\sum_{i=r_2+1}^n y_i) - T}{n - r_2}$. Assume, by contradiction, that $R_n^{\theta_1}(N, y, T) < R_n^{\theta_2}(N, y, T)$, i.e., $\mu_1 > \mu_2$. Then, $R_i^{\theta_1}(N, y, T) < R_i^{\theta_2}(N, y, T)$ for all $i = r_1 + 1, ..., n$. Finally, let $k \in \{r_2 + 1, ..., r_1 - 1\}$. Then, $R_k^{\theta_1}(N, y, T) = \theta_1 y_k \leq \theta_2 y_k \leq y_k - \mu_2 = R_k^{\theta_2}(N, y, T)$. Thus,

$$T = \sum_{i=1}^n R_i^{\theta_1}(N, y, T) < \sum_{i=1}^n R_i^{\theta_2}(N, y, T) = T,$$

which represents a contradiction.

Case 3: $\theta_1 Y < T < \theta_2 Y$.

By the definition of the TAL-family, $R_i^{\theta_1}(N, y, T) = \max\{\theta_1 y_i, y_i - \mu\}$ and $R_i^{\theta_2}(N, y, T) = \min\{\theta_2 y_i, \lambda\}$ for all $i \in N$, where μ and λ are chosen so as to achieve feasibility. Let r_1 be the smallest non-negative integer in $\{0, ..., n-1\}$ such that $T \geq \theta_1 Y + (1 - \theta_1)((\sum_{i=r_1+1}^{n} y_i) - (n - r_1)y_{r_1+1})$. Furthermore, let r_2 be the smallest non-negative integer in $\{0, ..., n-1\}$ such that $T \leq \theta_2((\sum_{i=1}^{r_2} y_i) + (n - r_2)y_{r_2+1})$. Thus,

$$
\begin{aligned}
R^{\theta_1}(N, y, T) &= (\theta_1 y_1, ..., \theta_1 y_{r_1}, y_{r_1+1} - \mu, ..., y_n - \mu), \text{ and} \\
R^{\theta_2}(N, y, T) &= (\theta_2 y_1, ..., \theta_2 y_{r_2}, \lambda, ..., \lambda),
\end{aligned}
$$

where $\lambda = \frac{T - \theta_2\left(\sum_{i=1}^{r_2} y_i\right)}{n - r_2}$ and $\mu = \frac{\theta_1\left(\sum_{i=1}^{r_1} y_i\right) + \left(\sum_{i=r_1+1}^{n} y_i\right) - T}{n - r_1}$. Consequently, $R_i^{\theta_1}(N, y, T) \leq R_i^{\theta_2}(N, y, T)$ for all $i = 1, ..., \min\{r_1, r_2\}$. Assume, by contradiction, that $R_n^{\theta_1}(N, y, T) < R_n^{\theta_2}(N, y, T)$, i.e., $y_n - \mu < \lambda$. Then, $R_i^{\theta_1}(N, y, T) < R_i^{\theta_2}(N, y, T)$ for all $i = \max\{r_1, r_2\}, ..., n$. Finally, let $k \in \{\min\{r_1, r_2\} + 1, ..., \max\{r_1, r_2\} - 1\}$.

If $r_1 < r_2$ then $R_k^{\theta_1}(N, y, T) = y_k - \mu \geq \theta_1 y_k$ whereas $R_k^{\theta_2}(N, y, T) = \theta_2 y_k \leq \lambda$. Thus,

$$
R_k^{\theta_1}(N, y, T) = y_k - \mu < y_k - y_n + \lambda \leq y_k - (1 - \theta_2)y_n \leq \theta_2 y_k = R_k^{\theta_2}(N, y, T).
$$

If $r_1 > r_2$ then $R_k^{\theta_1}(N, y, T) = \theta_1 y_k \geq y_k - \mu$ whereas $R_k^{\theta_2}(N, y, T) = \lambda \leq \theta_2 y_k$. Thus,

$$
R_k^{\theta_1}(N, y, T) = \theta_1 y_k \leq \theta_1 y_{r_1+1} \leq y_{r_1+1} - \mu < \lambda = R_k^{\theta_2}(N, y, T).
$$

We have therefore shown that, in any case, $R_k^{\theta_1}(N, y, T) < R_k^{\theta_2}(N, y, T)$ for all $k \in \{\min\{r_1, r_2\} + 1, ..., \max\{r_1, r_2\} - 1\}$. Thus,

$$
T = \sum_{i=1}^{n} R_i^{\theta_1}(N, y, T) < \sum_{i=1}^{n} R_i^{\theta_2}(N, y, T) = T,
$$

which represents a contradiction. ∎

References

[1] Aumann RJ, Maschler M, 1985, Game theoretic analysis of a bankruptcy problem from the Talmud, Journal of Economic Theory 36, 195-213.

[2] Becker, G., 1985, Public Policies, Pressure Groups and Dead Weight Costs, Journal of Public Economics 28, 329-347.

[3] Bénabou, R., 2000, Unequal Societies: Income Distribution and the Social Contract, American Economic Review 90, 96-129.

[4] Bénabou, R., 2002, Tax and Education Policy in a Heterogeneous Agent Economy: What Levels of Redistribution Maximize Growth and Efficiency? Econometrica 70, 481-517.

[5] Burman, L., Khitatrakun, S., Leiserson. G., Rohaly, J., Toder, E., Williams, B., 2008, An Updated Analysis of the 2008 Presidential Candidates' Tax Plans. Tax Policy Center. Urban Institute and Brookings Institution.

[6] Campbell, D., 1975, Income distribution under majority rule and alternative taxation criteria, Public Choice 22, 23-35.

[7] Ferraz, C., Finan, F., 2008, Exposing Corrupt Politicians: The Effects of Brazil's Publicly Released Audits on Electoral Outcomes. Quarterly Journal of Economics 123, 703-745

[8] Fleurbaey, M., 2006, Is commodity taxation unfair? Journal of Public Economics 90, 1765-1787

[9] Fleurbaey, M., Maniquet, F., 2006, Fair income tax, Review of Economic Studies 73, 55-83.

[10] Foley, D., 1967, Resource allocation and the public sector, Yale Economic Essays 7, 45-98.

[11] Gans, J. S., Smart, M., 1996, Majority voting with single-crossing preferences, Journal of Public Economics 59, 219-237.

[12] Gouveia, M., Oliver, D., 1996, Voting over flat taxes in an endowment economy, Economics Letters 50, 251-258

[13] Grossman, G., Helpman, E., 2001, Special interest politics, MIT Press, Cambridge, MA.

[14] Hamada, K., 1973, A Simple majority rule on the Distribution of Income, Journal of Economic Theory 6, 243-264.

[15] Le Breton, M., Moyes, P., Trannoy, A., 1996, Inequality Reducing Properties of Composite Taxation, Journal of Economic Theory 69, 71-103.

[16] Marhuenda F., Ortuño-Ortin, I., 1998, Income taxation, uncertainty and stability. Journal of Public Economics 67, 285-300.

[17] Mirrlees, J., 1971, An exploration in the theory of optimum income taxation, Review of Economic Studies 38, 175-208.

[18] Mitra, T., Ok, E., 1996, Personal income taxation and the principle of equal sacrifice revisited, International Economic Review 37, 925-948.

[19] Moreno-Ternero J, Villar A, 2006, The TAL-family of rules for bankruptcy problems. Social Choice and Welfare 27, 231-249.

[20] Ok, E., 1995, On the principle of equal sacrifice in income taxation, Journal of Public Economics 58, 453-467.

[21] Roberts, K.W.S., 1977, Voting over income tax schedules, Journal of Public Economics 8, 329-340.

[22] Romer, T., 1975, Individual welfare, majority voting, and the properties of a linear income tax, Journal of Public Economics 4, 163-186.

[23] Rundquist, B., Strom, G., Peters, J., 1977, Corrupt Politicians and Their Electoral Support: Some Experimental Observations, The American Political Science Review 71, 954-963.

[24] Thomson, W., 2008, Two families of rules for the adjudication of conflicting incomes. Social Choice and Welfare 31, 667-692.

[25] Young P., 1988, Distributive justice in taxation, Journal of Economic Theory 44, 321-335.